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PROPERTIES OF SOME NON-CONTINUOUS FUNCTIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "PROPERTIES OF SOME NON-CONTINUOUS FUNCTIONS", submitted by IVAN BAGGS in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

For the most part, this thesis represents an attempt to determine various properties of some non-continuous functions. In what follows, the real line (with its usual topology) is denoted R.

All functions in Chapter I are real valued functions of a real variable. In this chapter we are concerned with (a) the existence of open functions which are totally disconnecting, (b) the existence of open and closed functions which are discontinuous and (c) a characterization of closed sets of first category in terms of the points of discontinuity of a function with a closed graph. The main results are as follows:

(a) There exists an open function $f : R \to R$ such that f is totally disconnecting,

(b) Every real valued function of a real variable which is both open and closed is continuous,

(c) A set $B \subset R$ is closed and of first category in R if and only if there exists a function $f : R \to R$ such that f has a closed graph and the points of discontinuity of f coincide with B.

In Chapter II we first study the relationships between peripherally continuous, nearly continuous and connected functions. Secondly, we exhibit some properties of connectivity and almost continuous functions. The highlights of this chapter are as follows:

(a) Let X,Y be T_2 spaces, with X connected. Let $f : X \to Y$ be a connectivity function such that the set of points U, where f is continuous, is open and dense in X and $f|X - U$ is continuous. Then f cannot be redefined on $X - U$ such that f becomes

continuous on X .

(b) Let F be the family of connectivity functions from R into R . Then there exist f_1 and $f_2 \in F$ such that sup (f_1,f_2) has a totally disconnected graph.

(c) Let I to denote the closed unit interval. Let $f : I \to I$ be a connectivity function and let D_f be the set of points where f is discontinuous. If D_f is a closed set of first category and f is constant on D_f , then f is almost continuous.

In Chapter III we investigate for what spaces Y does there exist a connected (connectivity) function from the unit interval I onto Y ? For certain connected spaces (X,T) we will, with the aid of connectivity functions, construct a topology T^* on X such that T^* is strictly larger then T and (X,T^*) and (X,T) have the same connected sets. The main results of chapter III are as follows:

(a) A topological space Y with cardinality less than or equal to the cardinality of the continuum is connected if and only if Y is the connected image of the unit interval.

(b) Let X be connected and locally connected, Y second category and metric, and let f be a function from X into Y . If f is sequential and of Baire class I, then f has a connected graph.

(c) Let Y be any topological space such that $I \times Y$ is completely normal and the cardinality of the topology on $I \times Y$ is equal to the cardinality of the continuum. Then Y is a connectivity image of the unit interval.

(d) Let (X,T) be a first countable and connected Hausdorff space. Then there exists a topology T^* for X such that T^* is strictly larger then T and (X,T^*) and (X,T) have the same connected sets.

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INTRODUCTION - AN HISTORICAL NOTE

From the time one is introduced to beginning calculus one learns that the concept of a continuous function is fundamental to mathematics. However from beginning calculus onward one also encounters many important function which are not continuous.

This thesis represents an attempt to investigate some of the properties of various non-continuous functions. We concern ourselves mainly with open, closed, connected, connectivity, peripherally continuous and almost continuous functions, as well as a new class of functions which we have chosen to call sequentially continuous. It is however, the investigation of connectivity functions which occupies most of our efforts. (Definitions for the properties just mentioned, as well as those introduced below can be found at the beginning of the chapters into which the main body of this thesis is divided.)

The study of non-continuous functions is not a new topic in mathematics. Perhaps the derivative function was the first to receive concentrated attention. It was known during the last century that the derivative function has a connected graph. But this was not the only non-continuous function which interested mathematicians during the nineteenth century. For example, Cauchy showed in 1821 that a real valued function f of a real variable satisfying the condition that $f(x + y) = f(x) + f(y)$ is either continuous or everywhere discontinuous. It was not until 1905 that Hamel showed the existence of a discontinuous function satisfying this equation. In 1942 F. B. Jones [10] showed that the graph of such a function may be connected as a subset of the plane

but may still be everywhere discontinuous.

Early in this century Baire introduced a class of functions which now takes his name. This family includes the continuous functions. The characterization of the Baire functions has been an important part of the study of the Lebesgue integral. Darboux functions, which are so named because of the work of the famous French mathematician Darboux, have also played an important role in the mathematical analysis of this century. However, we will follow the convention of most modern writers and henceforth we will use the term connected functions rather than Darboux functions. Kuratowski and Sierpinski showed that a real valued Baire class I function of a real variable is a connected function, if and only if it has a connected graph.

Connectivity functions and some of the other non-continuous functions with which we will concern ourselves got a new start in the late 1950's with the work of Stallings [26] and Hamilton [5]. In 1957 Nash [21] defined a connectivity function from a space A into a space B as a mapping T such that the induced map g of A into $A \times B$ defined by $g(p) = (p,T(p))$ transforms connected sets of A onto connected sets of $A \times B$. He asked whether or not every connectivity function of a closed n-cell into itself has a fixed point. Hamilton [5] gave an affirmative answer to this question. In order to do this he introduced a new family of functions which he called peripherally continuous and showed that every peripherally continuous function from an n-cell into an m-cell $n,m \geq 2$, is a connectivity function. He then showed that every peripherally continuous function from I^n into I^m where $n \geq 2$ has a fixed point. So far the existence of an everywhere

discontinuous connectivity function from I^n onto I^m $n,m \geq 2$, has not been demonstrated. Hamilton has shown that there exists a connected function from I to I which has no fixed point.

Stallings [26] extended the work of Hamilton and introduced the concept of an almost continuous function. He also showed that if X is an Hausdorff space and if every continuous function $f : X \to X$ has a fixed point, then every almost continuous function $g : X \to X$ has a fixed point. At the end of his paper Stallings posed several questions, many of which have attracted the interest of others. The first one of these questions has remained unsolved and is given a partial solution in Chapter II of this thesis. He also asked if a connectivity function $f : I \to I$ is necessarily almost continuous. Several people, including Thomas [28], Roberts [23] and Brown [1] have since given an example to show that a connectivity function f from I to I is not necessarily almost continuous. Conversely it has been shown by Whyburn [30] and Hagan [6] that every peripherally continuous function defined on a connected, locally connected, locally peripherally connected and unichorent metric space into a space T such that $S \times T$ is completely normal, is a connectivity function. Thus, a function f from I^n into I^m, $n,m \geq 2$, is a connectivity function if and only if f is peripherally continuous.

Hilderbrand and Sanderson [7] have defined a connectivity retract and have shown that X is a finite polyhedron or has an ordered topology then every connectivity retract has a fixed point if every continuous function has a fixed point. They also showed that if f is a connectivity function and g is continuous then gf is also

a connectivity function. This is not true where g is only required to be connectivity even if f is continuous.

Cornette [2] showed that every separable and connected metric space is the connectivity image of the unit interval. One of more interesting results of this thesis shows that if Y is a topological space such that $I \times Y$ is completely normal and its topology has cardinality equal to the cardinality of the continuum, then Y is a connectivity image of the unit interval, I.

<u>NOTATION</u> The terminology of this thesis generally follows that of Kelley [11]. Throughout we will use I to denote the closed unit interval and $[a,b]$ will denote the closed interval with end points a and b. We have used (a,b) to denote both an open interval and a point in the plane; however, it will be made clear from the context which is being referred to.

If the continuum hypothesis is used in proving a particular proposition, then this will be indicated both immediately before stating the proposition and immediately following its use in the proof.

The various important definitions, lemmas, theorems and examples in the following three chapters are numbered consecutively within each section of each chapter. Reference to a numbered item without mention of the chapter or section is to the item of that number in the same section in which the reference is made.

CHAPTER I

A. OPEN AND TOTALLY DISCONNECTING REAL FUNCTIONS

It is well known that if X and Y are topological spaces there may exist functions $f : X \to Y$ such that f is open and discontinuous. Let R denote the real line with the usual topology. Spira [25] considered the following question, can a function $f : R \to R$ be open and not continuous? S. Marcus [19] showed that there exists a function $f : R \to R$ such that f is open and everywhere discontinuous. His function satisfied the equation $f(x + y) = f(x) + f(y)$ for all $x,y \in R$ and every connected set in the domain is taken to a connected set in the graph. He then asked the following questions:

1) Does there exist an open function $f : R \to R$ such that f is not connected?

2) If the answer to 1) is affirmative does there exist an open function $f : R \to R$ which does not have the connected property on any interval?

We will give an affirmative answer to both these questions. First the following definition:

I. <u>DEFINITION</u> Let X and Y be topological spaces. A function $f : X \to Y$ is <u>totally disconnecting</u> if it takes every connected set in X to a disconnected set in Y .

II. LEMMA Let $\{G_n\}_{n=1}^{\infty}$ be a base for the usual topology on R. Then there exists two sequences of perfect sets $\{E_n^1\}_{n=1}^{\infty}$, $\{E_n^2\}_{n=1}^{\infty}$ such that

(a) $E_n^i \subset G_n$ for $i = 1,2$ and $n = 1,2,\ldots$

(b) $E_n^i \cap E_n^j = \Phi$ for all $n \neq m$ and $i,j = 1,2$.

(c) E_n^i contains no interval for $i = 1,2$, $n = 1,2,\ldots$

Proof: We may assume that G_1 is an open interval. It follows that we can select 'Cantor like' perfect sets E_1^1 and E_1^2 such that (a) $E_1^i \subset G_1$ for $i = 1,2$, (b) $E_1^1 \cap E_1^2 = \Phi$ and (c) E_1^i contains no interval for $i = 1,2$. There exists an interval $I_2 \subset G_2$ such that $I_2 \cap (E_1^1 \cup E_1^2) = \Phi$. Then choose 'Cantor like' perfect sets E_2^1 and E_2^2 such that (a) $E_2^i \subset I_2$ for $i = 1,2$, (b) $E_2^1 \cap E_2^2 = \Phi$ and (c) E_2^i contains no interval, $i = 1,2$.

Assume for all $k < n$ we have chosen E_k^1 and E_k^2 which satisfy the hypothesis of the theorem. Now consider G_n. There exists an interval $I_n \subset G_n$ such that $I_n \cap (\bigcup_{k=1}^{n-1}\{E_k^1 \cup E_k^2\}) = \Phi$. If no such interval I_n exists, then G_n is contained in $\bigcup_{k=1}^{n-1}\{E_k^1 \cup E_k^2\}$. However, since each E_k^j, $j = 1,2$, $1 \leq k < n$, is a perfect set containing no interval this would contradict the Baire category theorem. Hence there exists an interval $I_n \subset G_n$ with the required property. Then choose two perfect sets E_n^1 and E_n^2 such that (a) $E_n^i \subset I_n$ for $i = 1,2$. (b) $E_n^1 \cap E_n^2 = \Phi$ and (c) E_n^i contains no interval for $i = 1,2$. Hence by induction there exists a family of sets $\{E_n^1\}_{n=1}^{\infty}$, $\{E_n^2\}_{n=1}^{\infty}$ which satisfy the conditions of the lemma.

III. THEOREM There exists a function $f : R \to R$ such that f is open and totally disconnecting.

Proof: Let $\{G_n\}_{n=1}^{\infty}$ be a base for the usual topology on R. Let $\{E_n^1\}_{n=1}^{\infty}$, $\{E_n^2\}_{n=1}^{\infty}$ be the sequence of perfect sets constructed in the preceding lemma. Put $E^1 = \bigcup_{n=1}^{\infty} E_n^1$ and $E^2 = \bigcup_{n=1}^{\infty} E_n^2$. It follows from the construction of $\{E_n^i\}_{n=1}^{\infty}$, $i = 1,2$, that $E^1 \cap E^2 = \Phi$. Now Halperin [4] has constructed a function $g : R \to R$ such that g takes on every real value 2^{N_o} times on every perfect subset of R. Let h_1 be an homeomorphism of R onto $(0,1)$. Let h_2 be an homeomorphism of R onto $(2,3)$. Define a function f as follows:

$$f(x) = h_1(g(x)) \qquad \text{for } x \in E^1$$

$$f(x) = h_2(g(x)) \qquad \text{for } x \in E^2$$

$$f(x) = h_1(g(x)) \qquad \text{for } x \in (E^1 \cup E^2)^c .$$

Then f is well defined. We will now show that f is totally disconnecting. Let G be an open set in R. Then there exists some G_n in the base $\{G_n\}_{n=1}^{\infty}$ such that $G_n \subset G$. Hence by II there exists $E_n^1 \subset E^1$ and $E_n^2 \subset E^2$ such that E_n^1 and $E_n^2 \subset G$. Then since g maps every perfect subset of R onto R, $f(E_n^1) = (0,1)$ and $f(E_n^2) = (2,3)$. Hence $f(G) = (0,1) \cup (2,3)$. So f is open. Since every connected subset C of R contains an open set G, it follows that $f(C) = (0,1) \cup (2,3)$ and f is totally disconnecting.

From the preceding theorem it follows that we have an affirmative answer to the second question of the introduction to this section. It therefore follows that the first question is also answered in the

affirmative.

COROLLARY Let A and B be any two disjoint, non-degenerate, open subsets of R. Then there exists an open and totally disconnecting function $f : R \to R$ such that if V is any superset of an open set then $f(V) = A \cup B$.

Proof: This follows from the construction used in the proof of III.

In the next theorem we will make use of the following result due to C. Kuratowski and W. Sierpinski [15].

IV. THEOREM Every real perfect set contains 2^{N_o} disjoint perfect sets.

We now show that there exists a "large" family of open and totally disconnecting functions from R into R. The next theorem makes use of the continuum hypothesis.

V. THEOREM There exists 2^{N_o} distinct functions from R into R such that each is open and totally disconnecting.

Proof: Let $\{G_n\}_{n=1}^{\infty}$ be a base for the usual topology on R and for each n select E_n^1 and E_n^2 such that $E_n^1 \cup E_n^2 \subset G_n$ as outlined in the proof of II. Let Ω_o be the first ordinal number with uncountably many predecessors. For some fixed integer n, let $\{E_\alpha^i\}_{\alpha \leqq \Omega_o}$ be a collection of perfect sets such that for $\alpha \neq \beta$, $\alpha, \beta < \Omega_o$, $E_\alpha^i \cap E_\beta^i = \Phi$ and $E_\alpha^i \subset E_n^i$ for $\alpha \leqq \Omega_o$ and $i = 1,2$. For each α put $A_\alpha^1 = (\bigcup_{\substack{j=1 \\ j\neq n}}^{\infty} E_j^1) \cup E_\alpha^1$ and $A_\alpha^2 = (\bigcup_{\substack{j=1 \\ j\neq n}}^{\infty} E_j^2) \cup E_\alpha^2$. It is clear

that $A_\alpha^2 \cap A_\alpha^1 = \Phi$ for all $\alpha \leqq \Omega_o$ and for every open set G_n of the base there exists perfect sets A_n^1 and A_n^2 such that $A_n^i \subsetneq A_\alpha^i$ $i = 1,2$ $A_n^1 \cup A_n^2 \subset G_n$. Let g be the function defined by Halperin [4] such that $g : R \to R$ and g takes on every real value 2^{N_o} times on every perfect subset of R . Let h_1 be a homeomorphism of R onto $(0,1)$ and h_2 a homeomorphism of R onto $(2,3)$. Define a function f_α as follows for each fixed $\alpha \leqq \Omega_o$:

$$f_\alpha(x) = h_1(g(x)) \qquad x \notin A_\alpha^2$$
$$f_\alpha(x) = h_2(g(x)) \qquad x \in A_\alpha^2$$

Then f_α is well defined and it follows as in the proof of theorem III that each $f_\alpha, \alpha \leqq \Omega_o$, is open and totally disconnecting. It is clear from the construction of each f_α , that if $\alpha \neq \beta$, then since $A_\alpha^i \neq A_\beta^i$, $i = 1,2$, $f_\alpha \neq f_\beta$, $\alpha,\beta \leqq \Omega_o$. Then by assuming the continuum hypothesis, that is, that the cardinality of the set of ordinals less then or equal to the first uncountable ordinal is 2^{N_o}, we achieve the required result.

It is immediate from the foregoing results that an open function may not take connected sets to connected sets. Even if we require the function to satisfy stronger conditions it may still not be continuous. For example Marcus [18] showed that there exists a function f from R onto R such that $f(x + y) = f(x) + f(y)$ for all x and y in R and f is everywhere discontinuous. Actually such a function may have a connected graph and still not be continuous.

Question: Does there exist a function $f : R \to R$ such that f is open,

satisfies the equation $f(x + y) = f(x) + f(y)$ for all $x,y \in R$ and is totally disconnecting?

To conclude this section we present one more result concerning open and everywhere discontinuous functions.

VI. THEOREM There exists a function $f : R^n \to R^m$, where $n,m \geq 1$, such that f is onto, open and everywhere discontinuous.

Proof: Let $f_1 : R^n \to R_1$ denote the projection mapping. Then f_1 is open and onto. Let $g : R^1 \to R^1$ be a function such that g is open and takes every real value on every perfect set. Let $h : R^1 \to R^n$ be such that h is one to one and onto, such a mapping exists since the cardinality of R^1 and R^m are identical. Put $f = h o g o f_1$. Then f is an open transformation from R^n onto R^m and f takes on every value $2^{\aleph_o}$ on every open subset of R^n. So f has the required properties.

B. OPEN AND CLOSED REAL VALUED FUNCTIONS OF A REAL VARIABLE

As was pointed out in the last section, there exists a function $f : R \to R$ such that f takes on every real value $2^{\aleph_o}$ times on every perfect set. Halperin [4] and Marcus [18] also showed that there exists a function f from R onto R where $f(x + y) = f(x) + f(y)$ for all $x,y \in R$ such that f takes on every real value $2^{\aleph_o}$ times on every perfect set. It therefore follows that there exists an open function $f : R \to R$ such that f assumes every real value on every open interval. It is interesting to note from the foregoing that there exists a function $f : R \to R$ such that $f(x + y) = f(x) + f(y)$ for all $x,y \in R$, f is

open, f takes closed intervals to a closed set and f maps all perfect sets to closed sets.

(a) Does there exist a function $f : R \to R$ such that f is closed and maps every perfect set onto R ?

In section A we showed the existence of open real functions which are totally disconnecting.

(b) Does there exist an open and closed real valued function of a real variable which is both open and closed and totally disconnecting?

We will give a negative answer to both these questions.

I. LEMMA Let $x \in R$ and $f : R \to R$. If there exists an interval I containing x as an interior point such that every non-degenerate closed subinterval of I containing x maps onto R, then f is not closed.

Proof: Suppose $f : R \to R$ is a closed function and for some $x \in R$ and some interval I containing x as an interior point, every closed subinterval I_x of I containing x is mapped onto R . Then for every positive integer n let $S_{1/n}(x)$ denote the non-degenerate closed interval with center x and radius $\frac{1}{n}$ such that $S_{1/n}(x) \subseteq I$. Let $y \in R$ such that $f(x) \neq y$. Since for each n, $f(S_{1/n}(x)) = R$, we may select $y_n \in f(S_{1/n}(x))$ such that $y_n \neq y$ for $n = 1,2,...$ and $|y - y_n| < \frac{1}{n}$. Then choose an arbitrary element $x_n^1 \in f^{-1}(y_n) \cap S_{1/n}(x)$ and it follows that $x_n^1 \to x$. Hence $\{x_n^1\}_{n=1}^{\infty} \cup \{x\}$ is a closed

set in R . But $f(\{x_n^1\}_{n=1}^{\infty} \cup \{x\})$ is not closed in R since y is a limit point of this set but not a point of the set which contradicts the assumption that f is a closed function.

COROLLARY There does not exist a function $f : R \to R$ such that f maps every perfect set onto R and f maps closed sets to closed sets.

Proof Follows from the preceding lemma.

Hence we have a negative answer to question (a) stated in the introduction to this section. The purpose of the next two lemmas and the subsequent theorem is to give a negative answer to question (b) stated in the introduction to this section. The next lemma shows that a function $f : R \to R$ which is both open and closed cannot map every closed interval to a non-compact interval.

II. LEMMA Let $x \in R$ and $f : R \to R$. If f is both open and closed then there exists some closed neighbourhood F of x such that if B is a compact subset of F containing x , then f(B) is compact.

Proof: Suppose there exists some $x \in R$ such that for every neighbourhood N of x , there exists a closed neighbourhood U of x , where $U \subset N$, and f(U) is not compact. Hence, since f is a closed function, f(U) is an unbounded closed set. We may therefore assume that every closed neighbourhood of x is mapped onto an unbounded set. For $n = 1,2,\ldots$, let S_n be an open interval of radius $\frac{1}{n}$ and center at x . We will first show that $f(\overline{S}_n)$ properly contains a ray of the form $[f(x), +\infty)$ or $(-\infty, f(x)]$ for $n = 1,2,\ldots$.

If $f(S_n)$ is connected, by assumption $f(\overline{S}_n)$ is unbounded and hence contains a ray of the type required. So suppose for some n ,

$f(S_n)$ is not connected. Then $f(S_n)$ can be written as the union of two disjoint open intervals. This follows since f is both an open and closed function and $f(\overline{S}_n) = f(S_n) \cup f(a_n) \cup f(b_n)$, where a_n , b_n are the end points of the open interval S_n . Without loss of generality, we may assume $f(S_n) = (a_1, a_2) \cup (b_2, +\infty)$. If there exists some integer N such that $N < a_1$, then since $f(\overline{S}_n)$ is closed, $a_2 = b_2$ and $f(\overline{S}_n)$ contains $[f(x), +\infty)$. If $a_1 = -\infty$, then $f(\overline{S}_n) = (-\infty, a_2) \cup (b_2, +\infty)$ and again we have $f(\overline{S}_n)$ containing $(-\infty, f(x)]$ or $[f(x), +\infty)$. We may therefore assume $f(\overline{S}_n) \supset [f(x), +\infty)$ for $n = 1,2,\ldots$.

Let $y \in [f(x), +\infty)$ such that $y \neq f(x)$. For each positive integer n choose $y_n \in [f(x), +\infty)$ such that $y_n \neq f(x)$ and $|y_n - y| < \frac{1}{n}$. Select $x_n \in S_n$ such that $f(x_n) = y_n$, for $n = 1,2,\ldots$. Then $x_n \to x$ and $\{x_n | n = 1,2,\ldots\} \cup \{x\}$ is a closed set in R . However $A = f(\{x_n | n = 1,2,\ldots\} \cup \{x\})$ has a limit point y , which is not contained in A . So A is not closed, contradicting the fact that f is a closed function. Hence there must exist some closed neighbourhood F of x such that every compact subset of F containing x is mapped onto a compact set.

Before stating the next lemma we need a theorem of Klee and Utz [13] which gives us a situation when a function which preserves connected sets and compact sets is a continuous function.

III. <u>THEOREM</u> (Klee and Utz [13] theorem B). Let $f : X \to Y$ where X is locally connected and X and Y are metric spaces. If f takes connected sets to connected sets and compact sets to compact sets, then f is continuous.

We now use this theorem to show that open and closed functions from R to R must have points of continuity.

IV. LEMMA Let $f : R \to R$ be an open and closed mapping. If there exists an open and bounded interval V such that $f(V)$ is bounded, then f is continuous at x for all $x \in V$.

Proof: Let I be an arbitrary interval contained in V . We will show that $f(I)$ is connected. Suppose, on the contrary, that $f(I) = A_1 \cup A_2$ where $A_1 = f(I) \cap V_1$ and $A_2 = f(I) \cap V_2$, V_1 and V_2 are disjoint open sets in R . Then $f(\text{int } I) \subseteq A_1 \cup A_2$ and is an open set in R . $f(\overline{I}) = f(\text{int } I) \cup f(a) \cup f(b)$, where a and b are endpoints of I , is closed in R . But since $f(I)$ is bounded, it follows that $f(I)$ must be an interval. Therefore f restricted to V is a connected function. Also, since f takes closed sets to closed sets, f restricted to V takes compact sets to compact sets. Therefore by the theorem of Klee and Utz stated above, the restriction of f to V is a continuous function. Since V is open in R , this implies f is continuous at x for every $x \in V$.

V. THEOREM Let $f : R \to R$ be an open and closed function, then f is continuous.

Proof: Let $x \in R$. By II there exists a closed and hence an open neighbourhood V of x such that f is bounded on V . Hence by III f is continuous at x . Since x was arbitrary, the result follows.

It is clear that V gives a negative answer to question (b) which was stated at the beginning of this section.

C. DISCONTINUOUS FUNCTIONS WITH A CLOSED GRAPH

Let f be a mapping of a Hausdorff topological space X onto another Hausdorff space Y. The subset $\{(x,f(x)) | x \in X\}$ of the product space $X \times Y$ is called the <u>graph</u> of f. If g is continuous, it is well known that the graph of f is a closed subset of $X \times Y$. It is also well known that the converse statement is not true even when $X = Y = R$. The following theorem which will be used in VII is also known (see Kolodner [12]).

I. <u>THEOREM</u> Let X and Y be Hausdorff spaces and let f be a mapping from X into Y. If the graph of f is compact as a subset of $X \times Y$, then f is continuous.

The main purpose of this section is to find a characterization of the points of discontinuity, D_f, of a real valued function of a real variable which has a closed graph. We consider the following questions:

(a) Can a function $f : R \to R$ have a point of discontinuity of the second kind?

(b) Given a function $f : R \to R$ which has a closed graph, what can we say about D_f?

Before answering these questions we need a few definitions and some notation.

II. <u>DEFINITION</u> Let $x \in R$ and consider the family of open intervals $(x, x + \frac{1}{n})$ $n = 1,2,\ldots$. Let h_n denote the least upper bound of f on

$(x, x + \frac{1}{n})$. Then the *upper limit of f at x from the right* is $\inf_n h_n$ and is denoted by $\overline{f(x+)}$.

Similarly we define the lower limit of f at x from the right and denote it by $\overline{f(x+)}$. The upper and lower limits of f at x from the left will be denoted by $\overline{f(x-)}$ and $\underline{f(x-)}$ respectively.

III. DEFINITION f has a *discontinuity at x of the first kind* if $\underline{f(x-)} = \overline{f(x-)}$, $\underline{f(x+)} = \overline{f(x+)}$ and at least one of these numbers fails to be equal to $f(x)$.

IV. DEFINITION If f is discontinuous at x and x is not a discontinuity of the first kind, then x is a point of *discontinuity of the second kind*.

It is easy to construct a function $f : R \to R$ such that f has a closed graph and has a point of discontinuity of the first kind. For example consider $f(x) = \frac{1}{x}$, $x \neq 0$ and $f(0) = 0$. Then f has a point of discontinuity of the first kind at 0 and the graph of f is closed in the plane. We now give an example of a function which has a closed graph and a point of discontinuity of the second kind.

EXAMPLE 1. Put $f(x) = 0$ for all $x \notin (0,1)$
and $f(x) = 0$ if $x = \frac{1}{n}$, $n = 1,2,\ldots$

Define f on the remaining intervals $(1/n + 1, 1/n)$, for $n = 1,2,\ldots,$ as follows; for $n = 1$,

$$f(x) = \frac{1/4}{x - 1/2} \qquad \text{if } x \in (1/2, 3/4] ,$$

$$f(x) = \frac{1/4}{1 - x} \qquad \text{if } x \in [3/4, 1) .$$

In general, for $n = 2,3,\ldots$ let m_n be the mid point of the interval $[1/n + 1, 1/n)$, and define

$$f(x) = \frac{n(m_n - \frac{1}{n+1})}{x - \frac{1}{n+1}} \qquad \text{if} \quad x \in (\frac{1}{n+1}, m_n] ,$$

$$f(x) = \frac{(\frac{1}{n} - m_n)}{\frac{1}{n} - x} \qquad \text{if} \quad x \in [m_n, \frac{1}{n}) .$$

It is readily verified that f is well-defined for all $x \in R$ and f is discontinuous at each point of D_f, where $D_f = \{0\} \cup \{\frac{1}{n} | n = 1,2,\ldots\}$. f has a closed graph, for let $(x,y) \in R \times R$ be a limit point of the graph of f, which is denoted by $\sqrt{f}$. If $(x,y) \notin \sqrt{f}$, then $x \neq \frac{1}{n}$, $n = 1,2,\ldots$. For if $x = \frac{1}{n}$ and $y \neq 0$ it follows from the definition of f that there exists a neighbourhood N of $(\frac{1}{n},y)$ such that $N \cap \sqrt{f} = \Phi$. Suppose $x = 0$, then y must also be 0 and if $(x_n,f(x_n)) \to (x,y)$, then $x_n \leq 0$ for all $n \geq N_0$. Since f is continuous at x from the right $y = f(x)$ and $(x,y) \in \sqrt{f}$. If $x \in \{0\} \cup \{\frac{1}{n} | n = 1,2,\ldots\}$, then there exists an open set containing x such that f is continuous at x and clearly (x,y) is a limit point of $\sqrt{f}$ if and only $(x,y) \in \sqrt{f}$. Hence the graph of f is closed. f has a discontinuity of the second kind at $x = 0$, for if $x_n = \frac{1}{n}$, $\lim_n f(x_n) = 0$ while if $x_n = m_n$ $n = 1,2,\ldots,$ then $m_n \to 0$ and $\lim_n f(m_n) = +\infty$, it is easily seen that $\underline{f(x+)} = 0$ while $\overline{f(x+)} = +\infty$.

If $f : R \to R$ is discontinuous at x, then not all the five quantities $f(x)$, $\overline{f(x+)}$, $\underline{f(x+)}$, $\underline{f(x-)}$ and $\overline{f(x-)}$ are equal. The oscillation function $w : R \to R \cup \{+\infty\}$ is defined at each $x \in R$ to

be the maximum of the difference between these quantities at x.

Hobson [8] gives the following result: "If $f(x)$ is defined on an interval, the oscillation function $w(x)$ is such that the set of points for which $w(x) \geq \alpha$, α a real number, forms a closed set".

V. LEMMA Let $f : R \to R$ be any function with a closed graph, then the set of points where f is discontinuous forms a closed set.

Proof: Let $x \in R$ be a point of discontinuity of f. Without loss of generality we may assume $\overline{f(x+)} \neq f(x)$. Assume $-\infty < p < +\infty$ and $\overline{f(x+)} = p$. Then (x,p) is a limit point of the graph of f, but by assumption $f(x) \neq p$, so $(x,p) \notin \sqrt{f}$, contradicting that the graph of f is closed. Hence $\overline{f(x+)} = +\infty$. Therefore if N is any arbitrary positive integer $w(x) > N$. Hence by the result quoted above the points of discontinuity of f form a closed set.

It is a consequence of V that if D_f is dense in any subinterval of R, then f is discontinuous on some closed subinterval.

VI. LEMMA If $f : R \to R$ has a closed graph and is bounded below, then f is lower semicontinuous.

Proof: Let $-\infty < M \leqq f(x)$ for all $x \in R$. Suppose there exists $x_o \in R$ such that f is not lower semicontinuous at x_o. Then given $\epsilon > 0$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x_o$ and $f(x_n) < f(x_o) - \epsilon$ for $n = 1,2,\ldots$ Since $\{(x_n, f(x_n)) | n = 1,2,\ldots\}$

is a bounded set in the plane, some subsequence $\{(x_{n_k}, f(x_{n_k}))\}_k^\infty$ converges, say to (x_o, p). But by the way the $\{x_n\}_{n=1}^\infty$ were choosen $f(x_o) \neq p$. This contradicts the fact that the graph of f is closed. Hence f is lower semicontinuous.

If f is a lower semicontinuous function which is bounded below, then it is easily seen that f may not have a closed graph. In fact, if f is a bounded function which has a closed graph, it follows that f is continuous. We will now give an example to show that "bounded below" cannot be omitted in lemma VI. In this example f will also have the property that $D_f = K$, where K is a Cantor subset of $[0,1]$ and every $x \in K$ is a point of discontinuity of f of the second kind.

Example II. Let K be the Cantor set in $[0,1]$ formed by removing the "middle one-third open intervals". Let I_n be the family of open intervals removed during the n^{th} step. Then $I_n = \{(a_k^n, b_k^n)\}_{k=1}^{2n-1}$ where $n = 1,2,3,\ldots$. Define $f : R \to R$ such that $f(x) = 0$ if $x \in \{[0,1]^c \cup K\}$. If n is even, define f on I_n as follows:

$$f(x) = n\frac{(m_k^n - a_k^n)}{x - a_k^n} \qquad \text{if} \quad x \in (a_k^n, m_k^n]$$

$$f(x) = n\frac{(b_k - m_k)}{b_k - x} \qquad \text{if} \quad x \in [m_k^n, b_k^n)$$

where m_k^n is the midpoint of (a_k^n, b_k^n) for $k = 1,2,\ldots 2^{n-1}$.

If n is odd define f on I_n as follows:

$$f(x) = -n\frac{(m_k^n - a_k^n)}{x - a_k^n} \qquad \text{if} \quad x \in (a_k^n, m_k^n]$$

$$f(x) = -n\frac{(b_k - m_k)}{b_k - x} \qquad \text{if} \quad x \in [m_k^n, b_k^n)$$

where m_k^n is the mid point of (a_k^n, b_k^n) for $k = 1,2,\ldots 2^{n-1}$.

It is immediate that f is well defined, continuous on K^c and discontinuous at each point of K . It is also seen from the definition of f that if $x \in K$ then $\underline{f(x+)} = \underline{f(x-)} = -\infty$ and $\overline{f(x-)} = \overline{f(x+)} = +\infty$, so at each point of K, f has a discontinuity of the second kind. Let $x_n \to x$ be such that $f(x_n) \to p$, p finite. Then by the construction $p = f(x)$ and so the graph of f is closed. Also if $x \in K$, f is neither upper nor lower semicontinuous at x , since $\overline{f(x+)} = +\infty$ and $\underline{f(x-)} = -\infty$.

The above example shows that a function with a closed graph can have a discontinuity of the second kind at each point of a perfect subset of R which is of first category in R . We will now state the main results of this section which characterize a closed set of first category in R(that is, a closed nowhere dense subset of R) in terms of the points of discontinuity of a function with a closed graph. The example given in II will be shown to exemplify the "best type" of result possible. To be more precise, every real valued function of a real variable which has a closed graph is continuous except at the points of a closed nowhere dense subset of R .

VII. <u>THEOREM</u> If $f : R \to R$ is a function with a closed graph, then the set D_f of points of discontinuity of f is a closed and nowhere dense set.

<u>Proof</u>: We have already seen in V that D_f is a closed subset of

R . Suppose D_f is dense in some interval I . Then, since D_f is closed, $D_f \supset I$. Now the set $B_n = \{x \in I \mid -n \leq f(x) \leq n\}$ is closed in I , for each positive integer n , since f has a closed graph and $I = \bigcup_{n=1}^{\infty} B_n$. Hence some B_n contains a subinterval J of I , since I is of second category. But then f is a bounded function with a closed graph on J and hence by I, f is continuous on J , which is impossible since $J \subset I \subset D_f$.

VIII. THEOREM Let B be a subset of R which is closed and nowhere dense in R . Then there exists a function $f : R \to R$ such that

(a) f is continuous on the complement of B ,

(b) f is discontinuous at each point of B ,

(c) f has a closed graph.

Proof: Since B is closed and nowhere dense in R , B^c is open and dense in R . Let $B^c = \bigcup_{n=1}^{\infty} G_n$ where each G_n is an open interval and if $n \neq m$ $G_n \cap G_m = \Phi$. Let $G_n = (a_n, b_n)$, for $n = 1,2,\ldots,$ and let m_n be the midpoint of (a_n, b_n) for each n . Now define a function $f : R \to R$ such that $f(m_n) = n$, for $n = 1,2,\ldots,$ and f is continuous, monotonically increasing on $[m_n, b_n)$ and is asymptotic to the line $x = b_n$. Similarly, f is defined on $(a_n, m_n]$ such that $f(m_n) = n$, f is continuous, monotonically decreasing and asymptotic to the line $x = a_n$. If $x \in B$, then put $f(x) = 0$. It follows that f is well defined, f is continuous on B^c and discontinuous on B , for if $x \in B$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in B^c such that $x_n \to x$ but $f(x_n) \not\to f(x)$. It is also clear that the graph of f is

closed.

We have thus answered both questions (a) and (b) raised in the introduction to this section.

COROLLARY A set B is closed and nowhere dense in R if and only if there exists a function $f : R \to R$ such that f has a closed graph and $D_f = B$.

Proof: This is an immediate consequence of VIII and the corollary to VII.

IX. THEOREM Let $f : R \to R$ have a closed graph. If f is discontinuous at some $x \in R$, then there exists an interval I_x with endpoint x such that $f(I_x)$ is not connected.

Proof: Suppose f is discontinuous at x from the right. Then without loss of generality we may assume that $\overline{f(x+)} = +\infty$. Suppose for every interval of the form $[x,\frac{1}{n}]$ $n = 1,2,\ldots,$ that $f([x,\frac{1}{n}])$ is connected. Then $f([x,\frac{1}{n}]) \supseteq [f(x),\infty)$ for each n . Let $p \in R$ such that $f(x) < p$ and let N be any neighbourhood of (x,p) in $R \times R$. Then for every n there exists an element $x_n^1 \in [x,\frac{1}{n}]$ such that $(x_n^1,f(x_n^1)) \in N$. Hence we may choose a sequence $x_n^1 \to x$ such that $(x_n^1,f(x_n^1)) \to (x,p)$. This contradicts the fact that the graph of f is closed. Hence for some n , $f([x,\frac{1}{n}])$ is not connected, put $I_x = [x,\frac{1}{n}]$ for this particular n .

COROLLARY If f is a derivative function and f has a closed graph, then f is continuous.

<u>Proof</u>: This follows immediately from the preceding theorem and the fact that every derivative function takes connected sets in the domain to connected sets in the graph.

If A is a closed subset of the plane such that the projection π_1 of A is an interval I and if to each $x \in I$ there is associated a unique $y \in \pi_2(A)$ it follows from VII that every function $f : I \to R$ whose graph is contained in A, must be continuous on some subinterval of I. We now ask the following question: if for each $x \in I$ we do not have a unique $y \in \pi_2(A)$, then must there exist some function $f : I \to R$ such that the graph of f is contained in A and f is continuous on some subinterval of I ? The following example gives a negative answer to this question.

Example III. This example is a modification of an example given on page 84 [24]. Let $I = [0,1]$ and let $\{r_n\}_{n=1}^{\infty}$ be the sequence of rationals in I. Let $\sum_{n=1}^{\infty} C_n$ be an absolutely convergent series of positive numbers. Define a function $g : I \to R$ as follows:

$$g(x) = \sum_{r_n \leq x} C_n \qquad \text{where} \quad 0 < x \leqq 1 .$$

The sum is to be understood as taken over all C_n such that $r_n \leqq x$. Put $g(0) = 0$. Then f is monotonically increasing and has a point of discontinuity from the left at every rational point, in fact if x is rational $g(x-) < g(x)$ and if $x = r_n$ then $g(x) - g(x-) = C_n, n=1,2,\ldots$ g is continuous at x if x is irrational. Let A be the closure of the graph of g in $I \times R$. Then if $x \in \pi_1(A)$ and x is rational there exists y_1 and $y_2 \in R$ such that (x,y_1),

$(x_1 y_2) \in A$, while if x is irrational there exists only one y such that $(x,y) \in A$. Every function f which is defined on I and whose graph is contained in A will be discontinuous at every rational point. Hence there does not exist a subinterval $I^1 \subset I$ such that f is continuous on I^1 .

CHAPTER II

A. NEARLY CONTINUOUS, PERIPHERALLY CONTINUOUS AND CONNECTED FUNCTIONS

We will first present definitions of the various functions which are studied in this chapter. Let X,Y be topological spaces and f a function from X into Y.

I. DEFINITION f is nearly continuous [9] at $x \in X$ if for each open set V in Y containing $f(x)$, $\overline{f^{-1}(V)}$ is a neighbourhood of x in X.

Some authors [9], [16] refer to this property as almost continuous but we use the term nearly continuous in order to avoid confusion with almost continuous functions as given in definition V.

II. DEFINITION f is peripherally continuous [4], [5] at $x \in X$ if for every neighbourhood U of x and every neighbourhood V of $f(x)$, there exists a neighbourhood G of x contained in U such that the boundary, $B(G)$, of G in mapped into V by f.

III. DEFINITION f is connected if for every connected set $C \subset X$, $f(C)$ is connected in Y.

IV. DEFINITION f is connectivity [21] if for every connected set $C \subset X$, $\{(x,f(x)|x \in C\}$ is connected in the graph of f.

V. DEFINITION f is almost continuous [26] if for every open set V in $X \times Y$ which contains the graph of f there exists a continuous

function $g : X \to Y$ such that the graph of g is contained in V .

For the remainder of this section we will point out some of the relationships between nearly continuous, peripherally continuous and connected functions. Most of these properties will be fairly elementary but they appear to be new. It was shown by Lin [16] that if $f : X \to Y$ is a mapping from a Baire space X to a second countable topological space Y , then f is nearly continuous on a dense subset of X .

VI. LEMMA If $f : I \to Y$ is nearly continuous at a point $x \in I$, then f is peripherally continuous at x .

Proof: Let $x \in I$ and U and V be neighbourhoods of x and $f(x)$ respectively. Since $\overline{f^{-1}(V)}$ is a neighbourhood of x , there exists $n \in N$ such that $S_{\frac{1}{n}}(x) \subset U$ and $S_{\frac{1}{n}}(x) \subset \overline{f^{-1}(V)}$, where $S_{\frac{1}{n}}(x)$ is an open interval with center x and radius $\frac{1}{n}$. If x is not a right end point of I , then there exists $x_1 \in S_{\frac{1}{n}}(x)$ such that $x_1 > x$ and $f(x_1) \in V$. For if such an x_1 does not exist, then $f(y) \in V^c$ for all $y \in (x, x + \frac{1}{n})$ and so $y \notin \overline{f^{-1}(V)}$ which contradicts the fact that $S_{\frac{1}{n}}(x) \subset \overline{f^{-1}(V)}$. Similarly there exists $x_2 < x$ such that $x_2 \in S_{\frac{1}{n}}(x)$ and $f(x_2) \in V$. Hence f is peripherally continuous at x .

COROLLARY If $f : I \to Y$, where Y is a second countable topological space, then f is peripherally continuous on a dense subset of I .

<u>Proof</u>: This follows from the preceding Lemma and the result by Lin which is stated above.

Whyburn [30] and Hagan [6] showed that every peripherally continuous function $f : I^n \to I^m$, $n,m \geq 2$ is a connectivity function. The following simple example shows that a nearly continuous function $f : I^2 \to I^2$ may be neither peripherally continuous nor connected.

Put $f(x) = (0,0)$ if $x = (x_1,x_2)$ and x_1 and x_2 are rational

$f(x) = (1,1)$ otherwise.

It is also easy to see that a peripherally continuous function from I to I may not be nearly continuous. (Put $f(x) = \sin\frac{1}{x}$, $x \neq 0$ and $f(0) = 0$).

However the following result does hold:

VII. <u>THEOREM</u> Let X be any locally connected topological space and $f : X \to Y$ such that f is peripherally continuous. If the image of every closed set in X, with non empty interior, is dense in Y, then f is nearly continuous.

<u>Proof</u>: Suppose there exists $x \in X$ and a neighbourhood V of $f(x)$ such that for every neighbourhood U of x there exists an $x_1 \in U$ and a neighbourhood U_{x_1} such that $U_{x_1} \cap f^{-1}(V) = \Phi$. We may assume U_{x_1} is open and connected. Then $f(\overline{U}_{x_1}) \cap V \neq \Phi$. Hence the boundary of U_{x_1}, $B(U_{x_1}) \neq \phi$ and we may choose $y \in B(\overline{U}_{x_1})$ such that $f(y) \in V$. Since f is peripherally continuous every neighbourhood H of y contains an open neighbourhood H^1 such that the boundary of H^1 is mapped into V. But $H^1 \cap U_{x_1} \neq \Phi$ for all H^1 and since U_{x_1} is

connected $B(H^1) \cap U_{x_1} \neq \Phi$, i.e there exists some $y^1 \in U_{x_1}$ such that $f(y^1) \in V$. A contradiction, hence f is nearly continuous.

VIII. <u>THEOREM</u> If $f : I \to Y$ is connected, where Y is T_1, then f is peripherally continuous.

<u>Proof</u>: Assume false. Then for some $x \in I$, there exists a neighbourhood U and V of x and $f(x)$ respectively such that the boundary of every open set containing x and contained in U is not mapped into V. We may assume without loss of generality that for every $y \in U$ and $y \neq x$, the interval (x,y) is mapped into V^c. Then consider the image of $[x,y]$. Since $f(x)$ is closed and $f(x) \in V$, while $f((x,y]) \in V^c$, $f([x,y])$ is disconnected. A contradiction. Hence f is peripherally continuous.

If Y is not T_1 the above theorem may be false. The converse to the above theorem is also not true even when $Y = I$. To see this define $f : I \to I$ as follows: $f(x) = 0$ for x rational and $f(x) = 1$ otherwise. Then f is peripherally continuous but not connected. Also there exists a space X and an homeomorphism f from X onto X such that f is not peripherally continuous. For example let X be any set with more than one element. Let $x_o \in X$ be fixed. Define a topology T on X such that $U \in T$ if and only if U is a superset of x_o. Define $f : X \to X$ to be the identity mapping. For each $U \in T$ if $x \in U^c$, x is a boundary point of U. Hence f is not peripherally continuous.

It is easy to see that if X is any non regular topological space, then the identity function from X onto X is not peripherally

continuous. However if X is regular every continuous function from X into Y is peripherally continuous.

A one to one peripherally continuous function from I into I may not be connected. However:

IX. THEOREM If $f : I \to I$ is a one to one peripherally continuous function and if f^{-1} is connected, then f is a homeomorphism.

Proof: That f^{-1} is continuous follows from theorem 3.2 [22], and f is continuous by theorem 10 [17] .

B. CONNECTIVITY FUNCTIONS

In this section we begin the presentation of our results which indicate some of the properties of connectivity functions. First, it is clear that since the projection is a continuous function, every connectivity function is connected. The converse is not true even when $X = Y = R$ [22] . Solomon Marcus has shown that there exists a function $f : R \to R$ such that f assumes every real value 2^{N_o} times on every perfect set but f is not connectivity. Clearly f is connected. Our next theorem shows how disconnected the graph of a connected function can be.

I. THEOREM Let $f : R \to R$ be any connected function. If there exists a continuous function $g : R \to R$ such that $f(x) \neq g(x)$ for all x and if for every interval $I \subset R$ there exists x_I , $y_I \in I$ such that $f(x_I) < g(x_I)$ and $f(y_I) > g(y_I)$, then the graph of f is totally disconnected.

<u>Proof</u>: Let A be any subset of the graph of f, and let π_1 be the projection to the domain. If $\pi_1(A)$ is not an interval, then A is not connected in the product space. If $\pi_1(A)$ is an interval, the since the graph of g is closed, the graph of f restricted to A is not connected. Hence the graph of f is totally disconnected in $R \times R$.

We now use the last theorem to give an example which establishes the existence of a large family of connected functions from R to R such that each member of this family is connected, but has a totally disconnected graph.

<u>Example</u>: Let $h : R \to R$ be any function which assumes every real value 2^{N_o} times on every perfect set [4]. Then let $g : R \to R$ be any continuous function. Define f as follows:

$$f(x) = \begin{cases} 0 & \text{if } g(x) = h(x) \neq 0 \\ 1 & \text{if } g(x) = h(x) = 0 \\ h(x) & \text{otherwise} . \end{cases}$$

Then f assumes every real value on every perfect set. It follows from the preceding theorem that the graph of f is totally disconnected.

<u>COROLLARY</u> (Marcus [18] Theorem IV) There exists a real valued function defined on R which takes on every real value 2^{N_o} times on each perfect set and such that its graph is totally disconnected.

<u>Proof</u>: Follows from the preceding example.

Before stating a characterization of a function with a totally disconnected graph, we need the following notation and a

Theorem by Hildebrand and Sanderson [7] .

Let (X,T) and (Y,σ) be topological spaces and f a mapping from (X,T) into (Y,σ) . Put $g(x) = (x,f(x))$ for all $x \in X$. Then g will be referred to as the graph function of f .

Put $\sigma^1 = \{U \subset X | U = f^{-1}(V) , V \in \sigma\}$

Put $T^1 = T \vee \sigma^1$; that is, T^1 is the smallest topology containing both T and σ^1 .

Let $T \times \sigma$ denote the product topology on $X \times Y$. By a T-separation of a set $A \subset X$ we shall mean that $A = A_1 \cup A_2$ where A_1 and A_2 are open in A and $A_1 \cap A_2 = \phi$.

II. <u>LEMMA</u> (Hildebrand and Sanderson lemma 2.3 [7]) Given any function $f : (X,T) \to (Y,\sigma)$, if $K = A \cup B$ is a T^1-separation of a set $K \subset X$, then $g(K) = g(A) \cup g(B)$ is a $T \times \sigma$-separation of $g(K)$ (g is the graph function of f) .

III. <u>THEOREM</u> Let $f : (X,T) \to (Y,\sigma)$. The graph of f is totally disconnected if and only if T^1 is totally disconnected.

<u>Proof</u>: Suppose the graph of f is totally disconnected. If there exists a non singleton connected subset C in X , then since f is continuous in T^1 , the graph of f restricted to C is connected in $(T^1 \times \sigma) \supset T \times \sigma$ which implies that the graph of f restricted to C is connected in $T \times \sigma$. A contradiction. Hence the space (X,T^1) is totally disconnected.

Conversely, let A be a subset of the graph of f . If A

is connected in the $T \times \sigma$ topology, then $\pi_1(A)$ is a connected subset of X in the T^1 topology by lemma II. This implies that A is a singleton set.

The preceding theorem is analogous to theorem 2.4 of Hildebrand and Sanderson which says that "a function $f : (X,T) \to (Y,\sigma)$ is a connectivity function if and only if the connected sets of T and T^1 are the same".

IV. <u>THEOREM</u> Let (X,T) be any connected Hausdorff space and $f : X \to Y$ where (Y,σ) is also Hausdorff. Let $g : X \to Y$ be continuous. If $\{x \in X | f(x) = g(x)\}$ is an open and proper subset of X, then f is not a connectivity function.

<u>Proof</u>: Let $A = \{x \in X | f(x) \neq g(x)\}$. Let $x \in A$. To find a neighbourhood U_x of x in $T^1 = T \vee f^{-1}(\sigma)$ such that $U_x \subset A$. Since $f(x) \neq g(x)$, there exists $V_{f(x)}, V_{g(x)}$ such that $V_{f(x)} \cap V_{g(x)} = \phi$. Choose a neighbourhood G of x in T such that $g(G) \subset V_{g(x)}$. Then put $V_x = f^{-1}(V_{f(x)}) \cap G$. Let $y \in U_x$, to show $f(y) \neq g(y)$, this follows since $f(y) \in V_{f(x)}$ and $g(y) \in V_{g(x)}$. Hence U_x is the required neighbourhood. Since A is open in T^1 and $A^c = \{x | f(x) = g(x)\}$ is open in T it is also open in T^1, hence X is not connected with the T^1 topology. Therefore f is not a connectivity function by theorem 2.4 of Hildebrand and Sanderson.

When can a function f defined on a topological space X, such that $f|A$ is continuous, where A is a dense subset of X, be extended to a continuous function f_1 on X such that f_1 is continuous on X and $f_1|A = f|A$? This is an important and much written

about problem of general topology. Our next result gives a situation when f cannot be extended.

V. <u>THEOREM</u> Let X and Y be Hausdorff spaces and let X be connected. Let f be a connectivity function such that the points U, where f is continuous is a dense open subset of X and let $f|X - U$ be continuous. Then f cannot be redefined on $X - U$ such that f becomes continuous on X.

<u>Proof</u>: Suppose there exists an extension of f to f_1 such that $f_1|U = f|U$ and f_1 is continuous on X. Then for every $x \in X - U$ $f_1(x) \neq f(x)$, for since U is dense in X and f is not continuous at x there exists a net $x_\alpha \in U$ such that $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. Since $f_1(x_\alpha) = f(x_\alpha)$ for all α, and $f_1(x_\alpha) \to f_1(x)$, $f_1(x) \neq f(x)$ for all $x \in U$. Now applying IV and we have a contradiction to f being a connectivity function on X.

<u>COROLLARY</u> Let U,X,Y and f be as in the preceding theorem. In addition let U be connected, then if $U \subset A \subset X$, f cannot be redefined on $A - U$ such that f becomes continuous on A.

In the next chapter we shall show that there exists a large family of function with domain the unit interval such that each member of this family satisfies the conditions of theorem V.

Question: Can the condition that $f|X - U$ is continuous be weakened?

VI. <u>THEOREM</u> Suppose f and g are two lower semi-continuous functions from (X,T) to (R,σ), where σ is the usual topology on R,

X is connected and $f(x) \neq g(x)$ for all $x \in R$. If $T \vee f^{-1}(\sigma) = T \vee g^{-1}(\sigma)$, then neither f nor g is a connectivity function.

<u>Proof</u>: Put $A = \{x | f(x) > g(x)\}$

$B = \{x | f(x) < g(x)\}$.

Then $A \cap B = \Phi$. It is not difficult to show that $A \cup B$ is a separation of X in the $T \vee f^{-1}(\sigma)$ topology.

C. LATTICES OF CONNECTIVITY FUNCTIONS

If G is the family of continuous functions from a topological space X to the real line R, then G is a lattice under the operations of supremum and infimum. If F is the family of connectivity functions from X to R, is F a lattice under the same operations? We will show that even in the case where $X = R$, the answer to the above question is no. However we shall show that there exist an interesting family K of connectivity functions from R to R such that $G \subsetneq K \subsetneq F$ and K is a lattice. Using these results we will show that if $\{T_\alpha\}_{\alpha \in A}$ is a family of topologies on R such that for each $\alpha \in A$, $T_\alpha \supseteq T$, where T is the usual topology on R, and T_α has the same connected sets as T, then $T_\alpha \vee T_\beta$ may be a totally disconnected topology on R. However we shall show that there does exist a family $\{T_\beta\}_{\beta \in B}$ of topologies on R such that for each β, $T_\beta \supseteq T$, each T_β has the same connected sets as T and $\{T_\beta\}_{\beta \in B}$ is a lattice.

Before preceding with these results we need a theorem and one definition introduced by Brown [1].

I. DEFINITION (Brown [1]). Let f and g be the graphs of two functions from R into R. Then g cuts f if g has X-projection an interval and there are two points P and Q of f, P higher than Q, such that:

(a) the abscissa of P and Q are in the X-projection of g,

(b) every point of $cl(g)$ is lower then P and higher than Q,

(c) f and $cl(g)$ do not intersect.

Brown [1] then proves the following theorem which gives a sufficient condition for a real valued function of a real variable to be connectivity.

II. THEOREM If f is a function from R into R and no lower semi-continuous function cuts the graph of f, then f is a connectivity function.

Our next theorem shows that the family of connectivity functions from R into R is not closed under the operation of supremum and hence is not a lattice.

III. THEOREM Let F be the family of connectivity functions from R into R. Then there exist $f_1, f_2 \in F$ such that $f_1 \vee f_2$ has a totally disconnected graph $[(f_1 \vee f_2)(x) = \sup (f_1(x), f_2(x)]$.

Proof: Let $\{G_n\}_{n=1}^{\infty}$ be a family of open intervals which form a base for the usual topology on R. Select two perfect sets E_n^1 and E_n^2 such that (i) $E_n^1 \cup E_n^2 \subset G_n$, for $n = 1,2,\ldots$, (ii) $E_n^i \cap E_m^j = \Phi$, for $n \neq m$, $n,m = 1,2,\ldots,i,j = 1,2$ and (iii) E_n^i has empty interior,

for $i = 1,2$, $n = 1,2,\ldots$. This is possible by II section A of Chapter I .

Let $B = \{h : R \to R | h$ is of Baire class I$\}$. For each n , both B and E_n^1 have cardinality equal to the cardinality of the continuum. So there exists a one to one mapping $T_n : E_n^2 \to B$ such that T_n is onto for $n = 1,2,3,\ldots$. Define $f_1 : R \to R$ as follows:

$f_1(x) = f(x)$ for all $x \in E_n^1$, $n = 1,2,\ldots$, where f is any function from R to R which takes on every value $2^{\aleph_0}$ times on every perfect set.

(1) $f_1(x) = h(x)$ for all $x \in E_n^2$, $n = 1,2,\ldots$ where $T_n(x) = h \in B$.

If $x \in \bigcup_{n=1}^{\infty}(E_n^2 \cup E_n^1)^c$, then we define $f_1(x)$ arbitrarily but such that $f_1(x) \neq x$. On E_n^1 if $f_1(x) = x$ redefine $f_1(x)$ to be $x + 1$. Since every lower semi-continuous function is of Baire class I , it follows from the definition of f that if I is any interval and g any function of Baire class I , then there exists $x \in I$ such that $f_1(x) = g(x)$. It therefore is a consequence II that f_1 is a connectivity function.

Now define a second connectivity function f_2 from R into R as follows:

Put $f_2(x) = f(x)$, if $f(x) \neq x$, for $x \in E_n^2$, $n = 1,2,\ldots$, where f is the function referred to in (1) . If $f(x) = x$, then put $f_2(x) = x + 1$.

For each E_n^1 , $n = 1,2,\ldots$ there exists a set $E_n \subset E_n^1$ such that each E_n has cardinality equal to the cardinality of the continuum and $f_1(x) > x$ for all $x \in E_n$. There exists a one-to-one function

$L_n : E_n \to B$ such that each L_n is onto for $n = 1,2,\ldots$. Define f_2 on E_n such that for all $x \in E_n$, $f_2(x) = h(x)$ where $T_n(x) = h$, $n = 1,2,\ldots$. Define f_2 on $E_n^1 - E_n$ such that there exists an x_1 and y_1 where $f_2(x_1) < x_1$ and $f_2(y_1) > y_1$. Otherwise, arbitrarily for $y \in E_n^1 - E_n$ for $n = 1,2,\ldots$ and $x \in \bigcup_{n=1}^{\infty}(E_n^1 \cup E_n^2)^c$ just making sure that for all such x , $f(x) \neq x$. Again, applying II, we see that f_2 is a connectivity function.

Let $f(x) = f_1(x) \vee f_2(x)$ for all $x \in R$. It follows from the definition of f_2 on $E_n^1 - E_n$, for $n = 1,2,\ldots$, that given any interval $I \subset R$ there exists some $x \in I$ such that $f(x) < x$ and some $y \in I$ such that $f(y) > y$. Also for all $x \in R, f(x) \neq x$. Since $g(x) = x$ is a continuous function, it follows from I section B that the graph of f is totally disconnected.

However we shall see shortly that there do exist two discontinuous connectivity functions f_1, f_2 mapping R into R such that $f_1 \vee f_2$ is a connectivity function.

We will now show that there exist two topologies T_1 and T_2 on R such that T_1 and T_2 are both strictly larger then T , the usual topology on R , and T_1 and T_2 have the same connected sets as T , however $(R, T_1 \vee T_2)$ is a totally disconnected topological space.

Let f_1 and f_2 be the two connectivity functions constructed in the last theorem. Put $T_{f_1} = f_1^{-1}(T)$ and $T_{f_2} = f_2^{-1}(T)$. Put $T_1 = T \vee T_{f_1}$ and $T_2 = T \vee T_{f_2}$. Since both f_1 and f_2 are totally

discontinuous connectivity functions, T_1 and T_2 have the same connected sets as T and each is finer than T.

IV. LEMMA $g = f_1 \vee f_2$ is a continuous function for the $T_{f_1} \vee T_{f_2}$ topology.

Proof: Let $x \in R$ and assume $g(x) = f_1(x) = f_2(x)$. Let U be an arbitrary neighbourhood of $g(x)$. Then there exists $V_x^1 \in T_{f_1}$ and $V_x^2 \in T_{f_2}$ such that each is a neighbourhood of x in the respective topologies and $f_1(V_x^1) \subset U$ and $f_2(V_x^2) \subset U$. Then $G = V_x^1 \cap V_x^2$ is open in $T_1 \vee T_2$ and $f_1(G) \subset U$ and $f_2(G) \subset U$, hence $g(G) \subset U$ and g is continuous at x. If $g(x) = f_1(x) \neq f_2(x)$, then $f_1(x) > f_2(x)$ and we may choose a neighbourhood $U_{f_1(x)}$ and $U_{f_2(x)}$ such that $U_{f_1(x)} \cap U_{f_2(x)} = \phi$. Then choose neighbourhoods $V_x^1 \in T_{f_1}$ and $V_x^2 \in T_{f_2}$ such that $f_1(V_x^1) \subset U_{f_1(x)}$ and $f_2(V_x^2) \subset U_{f_2(x)}$. Then $g(V_x^1 \cap V_x^2) \subset U_{f_1(x)}$ and g is continuous in the $T_{f_1} \vee T_{f_2}$ topology.

V. THEOREM $(R, T_1 \vee T_2)$ is a totally disconnected topological space.

Proof: Put $T_{f_1 \vee f_2} = (f_1 \vee f_2)^{-1}(T)$. Then it follows from the previous lemma that $T_{f_1 \vee f_2} \subset T_{f_1} \vee T_{f_2}$, since $T_{f_1 \vee f_2}$ is the smallest topolgoy on R in which $f_1 \vee f_2$ is a continuous function. Hence $T \vee T_{f_1 \vee f_2} \subset T \vee (T_{f_1} \vee T_{f_2}) = (T \vee T_{f_1}) \vee (T \vee T_{f_2}) = T_1 \vee T_2$. It was shown in III that $f_1 \vee f_2$ has a totally disconnected graph, hence by III of section B, $(R, T \vee T_{f_1 \vee f_2})$ is a totally disconnected topological space. Therefore $(R, T_1 \vee T_2)$ is totally disconnected.

It follows from the construction of T_1 and T_2 that each is a topology on R which has the same connected sets as T, the usual

topology on R , and both T_1 and T_2 are strictly larger then T . However the above theorem shows that R with the topology $T_1 \vee T_2$, which is the smallest topology containing both T_1 and T_2 , is a totally disconnected topological space.

It should perhaps be mentioned in passing that there exists topological spaces (X,T) such that if (Y,σ) is any other topological space then every connectivity function from (X,T) into (Y,σ) is continuous. Whether there exists a topology T^* on R such that every connectivity function from R into (Y,σ) is continuous is an open question. This of course is equivalent to the question asked by Thomas [22] "does there exist a topology T^* on R such that (R,T^*) is connected and if T^1 is any topology on R strictly larger then T^* then (R,T^1) is not connected?".

We now produce a family K of connectivity functions from R to R such that K is a lattice under the operations of supremum and infimum. First we state without proof a theorem whose proof is given in the next chapter.

VI. THEOREM Let $f : R \to R$. If f is Baire class I and peripherally continuous then f is a connectivity function.

VII. THEOREM Let K be a family of function from R to R which satisfy the following conditions;

(i) If $f \in K$ then f is Baire class I and peripherally continuous.

(ii) If $f, g \in K$ such that both are discontinuous at $x_o \in R$, then there exists a sequence $x_n \to x_o$ such that $f(x_n) \to f(x_o)$ and $g(x_n) \to g(x_o)$.

Then K is closed under the operations of infimum and supremum.

Proof: Let f and $g \in K$. Then there exists sequences of continuous function $\{f_n\}_{n=1}^{\infty}$, $\{g_m\}_{m=1}^{\infty}$ such that $f_n \to f$ and $g_n \to g$, where the convergence is pointwise. Hence $(f_n \vee g_n) \to (f \vee g)$ and so $f \vee g$ is Baire class I. To show $f \vee g$ is peripherally continuous at x for each $x \in R$. If $(f \vee g)(x) = f(x)$ say, then by (ii) we can always find some sequence $x_n \to x$ such that $(f \vee g)(x_n) = f(x_n)$ $n = 1,2,\ldots$. Hence $f \vee g$ is peripherally continuous. Now, apply VI, and we have the required result.

COROLLARY Let f and g be mapping from R into R. If f and g are Baire class I, peripherally continuous and have no point of discontinuity in common, then $f \vee g$ is a connectivity function.

Proof: Follows immediately from the preceding theorem.

Since every derivative function is a connectivity function, the family K of the preceding theorem will contain all the derivative functions that satisfy (ii) of VII.

If either of the assumption of condition (i) are omitted from the hypothesis of VII, then since every connectivity function is a connected function, it follows from III of section A and III that the conclusion may not hold. We now give an example which shows that condition (ii) cannot be omitted.

<u>EXAMPLE</u> Let $G = \{(\frac{1}{2n}, \frac{1}{2n-1})\}$, for $n = 1,2,\ldots$. Let $H = \{(\frac{1}{2n-1}, \frac{1}{2n-2})\}$, for $n = 1,2,\ldots$, where for each n, $(\frac{1}{n}, \frac{1}{n-1})$ is an open interval in R. Define $f : R \to R$ as follows:

$$f(x) = 0 , \quad \text{for } x \in (-\infty,0]$$

$$f(x) = 1 , \quad \text{for } x \in [1,+\infty)$$

$$\text{or } x \in H .$$

Let m_n be the mid point of $(\frac{1}{2n}, \frac{1}{2n-1})$, for $n = 1,2,\ldots$. For each n define f on $[\frac{1}{2n}, m_n]$ and $[m_n, \frac{1}{2n-1}]$ such that the graph of f is a straight line in the plane joining the points $(\frac{1}{2n}, 1)$ and $(m_n, 0)$ and a straight line joining the points $(m_n, 0)$ and $(\frac{1}{2n-1}, 1)$ respectively.

Define $g : R \to R$ such that $g(x) = f(x)$ for $x \notin (0,1)$. $f(x) = 1$ for all $x \in G$. Then define g on H analogously to how f was defined on G. It is clear from the definition of f and g that both are Baire class I and peripherally continuous, hence connectivity functions. However f and g do not satisfy conditions (ii) of theorem VI. $f \vee g$ has a discontinuity at 0 of the first kind and is clearly not a connectivity function.

IV. ALMOST CONTINUOUS AND CONNECTIVITY FUNCTIONS

At the end of his paper [26] J. Stallings says "one of the important questions left unresolved is, under what conditions a connectivity map of the unit interval into a space is almost continuous". The questions which are then raised relating to this problem have been considered in several papers but to the best of my knowledge this question has never been answered. In this section we shall first provide a partial

answer to this question (see V), then we shall state a theorem (see VI) which ties together the various functions considered in this chapter.

I. LEMMA If $f : I \to X$, where X is an arbitrary space, takes closed connected sets to connected sets, then f is a connected function.

Proof: Suppose $A \subset I$ is connected and $f(A) = A_1 \cup A_2$ is a separation. $A = (a,b)$, where endpoints may or may not be included. To find a closed interval $B \subset A$ such that $f(B)$ is not connected. Choose an interval $[x_1,x_2] = B$ such that $[x_1,x_2] \subset A$ and $f(x_1) \in A_1$ and $f(x_2) \in A_2$. Then $f([x_1,x_2])$ is not connected. A contradiction.

II. LEMMA Let f be any function defined on I^n, $n = 1,2,\ldots$, such that f takes closed connected sets to connected sets, then f takes open connected sets to connected sets.

Proof: Let A be an open connected set. If $f(A) = A_1 \cup A_2$ is a separation, let $x \in f^{-1}(A_1) \cap A$ and $x_2 \in f^{-1}(A_2) \cap A$. Since A is open and connected there exists a path $[x_1,x_2]$ in A joining x_1 and x_2. Then $f([x_1,x_2])$ is not connected. A contradiction.

III. LEMMA Let $f : I \to Y$, where $I \times Y$ is completely normal and Hausdorff, such that f is almost continuous, then f takes connected sets to connected sets.

Proof: It follows from a corollary by Stallings [26] page 261 that if A is closed a connected set in I, then the graph of $f|A$ is connected. Hence $f(A)$ is connected in Y. By lemma I, f is connected.

IV. THEOREM If $g : X \to Y$ is almost continuous, then $f : X \to X \times Y$,

where f is defined by $f(x) = (x,g(x))$, is almost continuous.

Proof: Let V be an arbitrary open set in $X \times (X \times Y)$ containing the graph of f . To find a continuous function $h : X \to X \times Y$ such that the graph of h is contained in V . Since V is open in $X \times (X \times Y)$, for each $(x,x,g(x))$ there exists $V(x,x,g(x)) \subset V$ such that for some V_x and $V_{(x,g(x))}$, $V_x \times V_{(x,g(x))} \subset V_{(x,x,g(x))}$. Then $P = \bigcup_{x \in X} V_{(x,g(x))}$ is an open set containing the graph of $g(x)$ in $X \times Y$. Hence there exists a continuous function $p : X \to Y$ such that the graph of p is contained in P . Put $h(x) = (x,p(x))$ for all $x \in I$. Then since $\bigcup_{x \in X} (V_x \times V_{(x,g(x))})$ is contained in V , the graph of h is contained in V .

COROLLARY If $f : I \to Y$ is almost continuous, where $I \times Y$ is completely normal and Hausdorff, then f is a connectivity function.

Proof: It follows from IV that $g : I \to I \times Y$ is almost continuous, where g is the graph of f . By III g is a connected function, hence f is connectivity.

V. THEOREM Let $f : I \to I$ be a connectivity function and let D_f , the set of points where f is discontinuous, be closed and nowhere dense in R . If f is constant on D_f , then f is almost continuous.

Proof: Let V be an open set in $I \times I$ such that the graph of f is contained in V . Let $x \in D_f$ and V_x be an open sphere with center $(x,f(x))$ such that $V_x \subset V$. Then there exists some $y \in I$ such that f is continuous at y and $(y,f(y)) \in V_x$. For if this were not the case, let p be any end point of some open interval G_p such that f

is continuous on G_p , f is discontinuous at p and $p \in \pi_1(V_x)$. Choose a closed sphere S_p about $(p,f(p))$ such that $S_p \subset V_x$. Then $S_p \cup V_x^c$ is a separation of the graph of f restricted to the connected set $G_p \cup p$. This contradicts the fact that f is a connectivity function. Hence there exists $y \in I$ with the required property.

Let x_1 be the greatest lower bound of D_f in I . Choose an open sphere V_{x_1} about $(x_1,f(x_1))$ such that $V_{x_1} \subset V$. Let y_1 be a point in $\pi_1(V_{x_1})$ such that f is continuous at y_1 , $y_1 > x_1$ and $(y_1,f(y_1)) \in V_{x_1}$. If there exists $0 \leqq p_1 < x_1$ such that $(p_1,f(p_1)) \in V_{x_1}$ then join $(p_1 f(p_1))$ and $(y_1,f(y_1))$ by a straight line L_1 lying in V_{x_1} . If no such p_1 exists, join $(x_1,f(x_1))$ and $(y_1,f(y_1))$ by a straight line L_1 lying in V_{x_1} . Then for every $t \in [p_1,y_1]$ define a function $g_1 : [p_1,y_1] \to [0,1]$ such that $g(t) = \ell_t$, where ℓ_t is the y-coordinate of the point of the intersection of the vertical line through t with L_1 .

By choice of y_1 , the set of all $x \in D_f$ such that $x > y_1$ is again a closed set in I . Let $x_2 = \text{glb}\{x | x \in D_f \text{ and } x > y_1\}$. Now choose an open sphere V_{x_2} containing $(x_2,f(x_2))$ such that $V_{x_2} \subset V$ and $\pi_2(V_{x_2})$ has left endpoint greater than or equal to y_1 . Now proceed as before and choose a point y_2 to the right of x_2 such that f is continuous at y_2 and $(y_2,f(y_2)) \in V_{x_2}$. Now choose p_2 such that for all x where $y_1 \leqq x \leqq p_2$ f is continuous at x . Then define $g_2 : [p_2,y_2] \to [0,1]$ similar to the definition of g_1 on $[p_1,y_1]$. Continue this process and choose x_n for $n = 3,4,\ldots$ until we have exhausted the points of discontinuity of f . This will require

at most a countable number of steps. Now define $g : I \to I$ by

$$g(x) = f(x) \quad \text{if} \quad x \notin [p_n, y_n] \quad \text{for any} \quad n .$$

$$g(x) = g_n(x) \quad \text{if} \quad x \in [p_n, y_n] .$$

Then g is well defined, continuous and the graph of g is contained in V.

COROLLARY Let f be a function from I to I such that the set of points D_f where f is discontinuous is a closed set of first category and f is constant on D_f. Then f is almost continuous if and only if f is a connectivity function.

Proof: Follows from the last theorem and the corollary to IV.

QUESTION: Is it true that every function $f : I \to I$ which is Baire class I and peripherally continuous is almost continuous?

The above corollary gives us, in a special case, a characterization of almost continuous functions from I to I in terms of connectivity functions. Thus we have a partial answer to the question of Stallings stated in the introduction to this section. Finally, we state a theorem which ties together the various functions considered in this chapter. Let N, P, C_{-2}, C_{-1}, A denote the family of nearly continuous, peripherally continuous, connected, connectivity and almost continuous functions respectively from I to I.

VI. THEOREM Let N, P, C_{-2}, C_{-1}, A be as defined above, then $A \subsetneq C_1 \subsetneq C_2 \subsetneq P$ and $P \subsetneq N$. However N is neither contained in

any of the other families.

The proof of this theorem is evident from what has gone before.

<u>QUESTION</u>: Does there exist an almost continuous function $f : I \to I$ such that f is everywhere discontinuous?

CHAPTER III

A. EXISTENCE OF CONNECTED FUNCTIONS

Arkhangelshii wrote that one of the "fundamental problems" of general topology is "under what circumstances can each space of a given class A be mapped onto a space of class B by means of a mapping belonging to class L." For example, it is well known that a Hausdorff space Y is a continuous image of the closed unit interval if and only if Y is a compact, connected and locally connected metric space. Instead of taking f to be continuous, if we assume f is a connected or connectivity function, then for what spaces Y does there exist a connected or a connectivity function f mapping the closed unit interval onto Y ? When Y has cardinality less than or equal to the cardinality of the continuum and Y is connected we will show that there exists a connected funtion from the closed unit interval onto Y .

I. <u>THEOREM</u> A topological space Y , of cardinality less than or equal to the cardinality of the continuum, is connected if and only if there exists a connected function form I onto Y .

<u>Proof</u>: Clearly, if there exists a connected function from I onto Y , then Y is connected.

Conversely, let Y be any connected space with cardinality less than or equal to the cardinality of the continuum. Let f be a function from R onto Y . Let g be a function from I onto R such that g maps every perfect subset of I onto R . It was pointed out at the beginning of Chapter I Section A that such a mapping g exists. Then put $h = f \circ g$ and h has the required properties.

In this chapter our main problem is to show the existence of certain types of connectivity functions from I onto Y, where Y is a suitably choosen topological space. If Y is a separable metric space, then Cornette [2] has shown that there exists a connectivity function f from I onto Y. However, the function constructed by Cornette is everywhere discontinuous. We therefore consider the following Arkhangelskii type question: What class of topological spaces can Y represent in order to ensure that there exists a connectivity function $f : I \to Y$ such that f is onto and f has a point of continuity on every interval? Subsequently, we will consider a much more general space Y and show the existence of a connectivity function from I onto Y.

Before attempting these two questions in Section C, we introduce a family of functions which we call sequential functions. These functions will be used in Section C to construct a connectivity function f from I onto Y, where Y is the union of an ascending sequence of Peano spaces, such that f has a point of continuity when restricted to each closed subset of I.

B. SEQUENTIAL FUNCTIONS

In this section we will always assume that X and Y are first countable topological spaces.

I. DEFINITION A function $f : X \to Y$ will be called <u>sequentially continuous</u> on X if for every open set $G \subsetneq X$ there exists $x \in X - G$ such that $x \in \overline{G}$ and a sequence $x_n \to x$, $x_n \in G$, for $n = 1,2,\ldots$ such that $f(x_n) \to f(x)$.

It follows from the above definition that sequentially continuous functions are defined only on connected spaces.

If f is sequentially continuous on a space X, then $f|G$, where G is an open connected subset of X, may not be sequentially continuous on G. This can be seen from the following example.

Example I: Let $f : I \to I$ be defined as follows:

$$f(x) = 0 , \quad \text{for} \quad 0 \leqq x < 3/4 ,$$

and

$$f(x) = 1 , \quad \text{for} \quad 3/4 \leqq x \leqq I .$$

Then it is easily seen that f is sequentially continuous on I. However, let f_1 be the restriction of f to the open connected set $(1/2, 1]$. If we put $U = (1/2, 3/4)$ we see there does not exist an $x \in \overline{U}$, $x \notin U$ and a sequence $x_n \in U$ such that $x_n \to x$ and $f_1(x_n) \to f_1(x)$. Hence f_1 is not sequentially continuous on the open and connected set $(1/2, 1]$.

We will now introduce a term for a function whose restrictions to open and connected sets is a sequentially continuous function.

II. <u>DEFINITION</u> A function $f : X \to Y$ is called <u>sequential</u> on X if for every open connected set $G \subset X$, the restriction of f to G is sequentially continuous.

It is clear that every sequential function is sequentially continuous if X is connected. It also follows that every continuous function is a sequential function. However the converse is not true even when $X = Y = I$.

III LEMMA If X is a locally connected and regular space, then every peripherally continuous function f from X into Y is sequential.

Proof: Let G be an open connected subset of X ; so G is locally connected. Let U be an open proper subset of G . Since every component of U is open in G , it is sufficient to assume U is connected. Then there exists $x \in G$ such that $x \in \overline{U}$ and $x \notin U$. Let $\{V_n\}_{n=1}^{\infty}$ be a neighbourhood base at x such that $V_n \supset V_{n+1}$, for $n = 1,2...$, and let H be a neighbourhood of $f(x)$. Since f is peripherally continuous and X is regular, we can choose open neighbourhoods B_n of x , $n = 1,2...$ such that $B_n \subset \overline{B}_{n+1} \subset V_n$, for each n and $f(F(B_n)) \subset H$, where $F(B_n)$ denotes the boundary of B_n . We may also assume that $V_n \cap U \neq U$, for $n = 1,2...$ Thus there exists $y_n \in F(B_n) \cap U$, for $n = 1,2...$, since U is connected; and also $y_n \to x$ and by construction we may assume $f(y_n) \to f(x)$, since Y is first countable. Hence f is a sequential function.

However, as we show below, the converse to the above lemma is not true even when $X = Y = I^2$. This will be of some importance to us later when we use these functions to construct a connectivity function. For it has been shown by many authors (See [6] and [30]) that every peripherally continuous function from I^n into I^m , where $n,m \geqq 2$, is a connectivity function.

EXAMPLE I. We define a sequential function $f : I^2 \to I^2$ which is not peripherally continuous as follows:

$$f(x) = (0,0) \quad \text{when } x = (x_1,x_2) \text{ and } x_1 \text{ and } x_2 \text{ are rational,}$$

$$f(x) = (1,1) \quad \text{otherwise.}$$

f is a sequential function. For let G be an open and connected subset of I^2. Let U be an open subset of G and let x be a limit point of U such that $x \notin U$. It follows from the definition of f that we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in U$ for each n, $x_n \to x$ and $f(x_n) = f(x)$, for $n = 1,2,\ldots$. Hence f is a sequential function. However f is not peripherally continuous since f is not a connected function and every peripherally continuous function from I^2 into I^2 is connectivity [6].

On the other hand, the following relation does hold.

IV <u>THEOREM</u> A function $f : I \to Y$ is a sequential function if and only if f is peripherally continuous.

<u>Proof</u>: It follows from the preceding lemma that if f is peripherally continuous, then f is a sequential function.

Conversely, suppose f is a sequential function. Let $x \in I$ and let U and V be neighbourhoods of x and $f(x)$ respectively (U an open interval). Then consider $[o,x) \cap U = G$, since G is open in U there exists $x_n \to x$ such that $x_n \in G$, for $n = 1,2,\ldots$, and $f(x_n) \to f(x)$. Similarly, since $B = (x,1] \cap U$ is open in U and f is a sequential function, there exists $y_n \in B$, for $n = 1,2,\ldots$, such that $y_n \to x$ and $f(y_n) \to f(x)$. Therefore if V is an open set containing $f(x)$ and H is any open set containing x, there exists an open set $H^1 \subset H$ such that $x \in H^1$ and f maps the boundary of H^1 into V. So f is peripherally continuous.

The motivation for sequentially continuous functions stems from the following:

V THEOREM Let $f : X \to Y$ be a function with a connected graph, where X and Y are connected. Then f is sequentially continuous.

Proof: Suppose not. Then there exists an open set $G \subset X$ such that for every $x \in \overline{G} - G$ and for every $x_n \to x$, where $x_n \in G$ for $n = 1,2,\ldots$, $f(x_n)$ does not converge to $f(x)$.

Put $A = G \times Y$. For every $x \in \overline{G} - G$ choose a neighbourhood $N_{(x,f(x))}$ of $(x,f(x))$ such that $N_{(x,f(x))} \cap \{(g,f(g)) | g \in G\} = \Phi$. This can be done since f maps every boundary point of G to a point which is not a limit point of $f(G)$. Put $B = \bigcup_{x \in \overline{G}-G} N_{(x,f(x))}$, then B contains no point of the graph of f restricted to G. Put $C = \bigcup_{x \in X-\overline{G}} (N_x \times Y)$, where $N_x \cap G = \Phi$. C is open and contains no point of the graph of f restricted to G. Let A_1 and B_1 denote the intersection of A and $B \cup C$, respectively with the graph of f. It follows that $A_1 \cap B_1 = \Phi$, A_1, B_1 are open and contain the graph of f. Which contradicts the fact that f has a connected graph. Hence f is a sequentially continuous function.

COROLLARY 1. If $f : X \to Y$ is a connectivity function where X and Y are connected, then f is a sequential function.

Proof: This follows since we may show as in the last theorem that the restriction of f to every open and connected set is sequentially continuous.

COROLLARY 2. Let $f : I \to Y$ be a connectivity function, then f is peripherally continuous.

Proof: This is an immediate consequence of III and the preceding corollary.

As stated at the beginning, throughout this section we are assuming that X and Y are first countable. However, first countability can be omitted in the statement of corollary 2.

In the preceding theorem we assumed the function had a connected graph. This theorem does not hold with "connected graph" replaced by a "connected function".

EXAMPLE 2. Let $X = I$ and Y be any set with two or more points. Fix $x_o \in Y$ and let $\{x_o\}$ be an open set. For every other $y \in Y$, the only open set containing y is the whole space. This defines a topology T on Y. Define a function $f : I \to Y$ as follows:

$$f(x) = y \neq x_o \ , \quad \text{if} \quad x \neq 1/2$$

$$f(x) = x_o \qquad , \quad \text{for} \quad x = 1/2$$

It is easily seen that f is a connected function but f is not a sequentially continuous function. Of course, f does not have a connected graph.

Before presenting the final results of this section we will state a theorem by M.K. Fort which will be used in VII.

VI THEOREM (Fort [3]). If $f_1, f_2, \ldots$ is a sequence of continuous functions on a topological space X into a metric space Y, which converge pointwise to a function g, then g is continuous except at the points of a set of the first category.

The next theorem is a generalization of a theorem by Kuratowski and Sierpinski which says that every connected function $f : I \to I$ which

is of Baire class I is a connectivity function. This theorem, as we will state it, given us a link between functions with a connected graph and sequential functions. This result will be applied in the next section to construct connectivity functions which have points of continuity. For the remainder of this chapter we will understand that a Baire Class I function from a topological space X to a space Y is a function which is the point wise limit of a sequence of a continuous functions from X into Y.

VII THEOREM Let $f : X \to Y$ be a function, where Y is metric, X is connected and locally connected and every closed subset of X is of second category in itself. If f is sequential and Baire Class I, then f has a connected graph.

Proof: Let g denote the graph of f. Suppose g is not connected, then $g = A_1 \cup A_2$, where A_1, A_2 are open in g and $A_1 \cap A_2 = \Phi$. Let π_1 and π_2 be the projection mappings from $X \times Y$ onto X and Y respectively. Put $B_1 = \pi_1(A_1)$, $B_2 = \pi_2(A_2)$, $G_1 = \text{int}\,(B_1)$ and $G_2 = \text{int}\,(B_2)$.

Let $F_1 = \{x | x \epsilon X, N_x \cap B_1 \neq \Phi$, for every neighbourhood of $x\}$ and $F_2 = \{x | x \epsilon X, N_x \cap B_2 \neq \Phi$, for every neighbourhood of $x\}$.

Suppose p is a limit of $F_1 \cap F_2$; then every neighbourhood of p must intersect both B_1 and B_2, so $p \in F_1 \cap F_2$ and hence $F_1 \cap F_2$ is closed in X. Since f is of Baire Class I, $f|F_1 \cap F_2$ has a point of continuity by VI; say $x \in B_1$ is this point. Let $F = F_1 \cap F_2$. If there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $B_2 \cap F$ such that $x_n \to x$, then since f is continuous at x, $(x_n, f(x_n)) \to (x, f(x))$. But $(x, f(x)) \in A_1$ and $(x_n, f(x_n)) \in A_2$ and this would contradict the seperation of the graph of f by A_1 and A_2.

Hence there must exist some neighbourhood of x, say U_x, such that $U_x \cap (B_2 \cap F) = \Phi$. By the definition of F, it follows therefore, that every neighbourhood of x must intersect G_2. Let U be such an open and connected set such that $U \cap (B_2 \cap F) = \Phi$. Since f is a sequential function and U is open and connected, there exists $y \in U$ and a sequence $x_n \in 0 = U \cap G_2$ such that $x_n \to y$, $y \notin 0$ and $f(x_n) \to f(y)$. But since $y \notin G_2$, this implies that $y \in B_2 \cap F$, which contradicts the fact that $U \cap (B_2 \cap F) = \Phi$. Hence the graph of f is a connected set.

<u>COROLLARY</u> 1. Let $f : I \to Y$, where Y is a metric space and I the closed unit interval. Let f be a Baire Class I function. Then f is sequential if and only if f is a connectivity function.

<u>Proof</u>: Let f be a sequential function. It follows from the last theorem that the graph of f is connected. It then follows from a remark by Cornette [2] that f is a connectivity function.

Conversely let f be a connectivity function, then the fact that f is sequential follows from Corollary 1 to Theorem V.

<u>COROLLARY</u> 2. Let $f : I \to Y$, where Y is a metric space and I the unit interval. Let f be of Baire Class I, then f is peripherally continuous if and only if f is a connectivity function.

<u>Proof</u>: This is immediate since in this particular case sequential functions and peripherally continuous functions are equivalent (IV).

<u>COROLLARY</u> 3. Let $f : I \to Y$, where Y is a metric space and I the unit interval. Let f be of Baire Class I, then f is a connected

function if and only if f is a connectivity function.

Proof: This follows since every connected function is peripherally continuous and connectivity implies connected.

Corollary 3 is a generalization of a theorem by Kuratowski and Sierpinski which says that a Baire Class I real valued function of a real variable is a connected function if and only if it is a connectivity function.

C. EXISTENCE THEOREMS FOR CONNECTIVITY FUNCTIONS

In this section we will present two main results. First we will show the existence of a class B of spaces such that for every $Y \in B$ there exists a connectivity function from I onto Y with the property that f is continuous almost everywhere. Secondly, we will show the existence of a more general class A of spaces containing B such that for every $Y \in A$ there exists an everywhere discontinuous function f from I onto Y.

It is fairly easy to construct a connectivity function from I onto R such that f is continuous on $I - \{0\}$. Before proceeding to the more general case we present two special examples of connectivity functions from I onto R.

NOTATION: Given an interval $[a_{n+1}, b_n]$, where a_{n+1}, b_n are arbitary real numbers, let m denote the mid point of this interval. Let m_n denote the mid point of $[m, b_n]$ and let m_{n+1} denote the mid point of $[a_{n+1}, m]$. A real valued function f defined on $[a_{n+1}, b_n]$ with the

properties

(a) $f(a_{n+1}) = f(b_n) = f(m) = 0$,

(b) $f(m_{n+1}) = -n,\ f(m_n) = +n$,

and (c) f is linear on the intervals $[a_{n+1}, m_{n+1}]$, $[m_{n+1}, m_n]$ and $[m_n, b_n]$

will be denoted as $\surd\!\wedge$ - function of amplitude n .

<u>EXAMPLE</u> 1. We will show that if K is the Cantor subset in I , then there exists a function $f : I \to R$ such that f is onto, $D_f = K$ where D_f denotes the set of points where f is discontinuous, and f is a connectivity function.

Let $G_1 = \{(1/3,\ 2/3)\}$, $G_2 = \{(1/9,\ 2/9),\ (7/9,\ 8/9)\}$.

In general let $G_n = \{E_n^j\}_{j=1}^{2^{n-1}}$, where $E_n = (a_n^j\ ,\ b_n^j)$, for $n = 1,2,\ldots$ and $1 \leqq j \leqq 2^{n-1}$. That is, $\{G_n\}_{n=1}^{\infty}$ is the family of open sets which are removed from I to form the Cantor set. Let e_n^j be the mid point of E_n^j , $n = 1,2,\ldots$, $1 \leqq j \leqq 2^{n-1}$. Let $x_n^{j,1}$ be the mid point of $(a_n^j,\ e_n^j)$, for $n = 1,2,\ldots$, $j = 1,2,\ldots 2^{n-1}$. In general let $x_n^{j,k}$ be the mid point of $(a_n^j,\ x_n^{j,k-1})$, for $k = 2,3,\ldots$. Then $\{x_n^{j,k}\}_{k=1}^{\infty}$ is such that for each fixed j and each n , $\lim x_n^{j,k} = a_n^j$, for $n = 1,2,\ldots$ and $1 \leqq j \leqq 2^{n-1}$. In the same manner, for each fixed n and for each j such that $1 \leqq j \leqq 2^{n-1}$ construct a sequence $\{y_n^{j,k}\}_{k=1}$ such that $\lim\limits_{n} y^{j,k} = b_n^j$.

Define f_1 as follows: for $n = 1,2,\ldots$, $1 \leqq j \leqq 2^{n-1}$, $f_1(x) = g(x)$ if $x \in [x_n^{j,1},\ y_n^{j,1}]$, where g is a $\surd\!\wedge$ - function of amplitude 1. $f_1(x) = g(x)$, if $x \in [x_n^{j,2},\ x_n^{j,1}]$, where g is again a $\surd\!\wedge$ - function of amplitude 1. Define f_1 exactly the same way on

$[y_n^{j,1}, y_n^{j,2}]$. Put $f_1(x) = 0$ for all other $x \in [0,1]$. Then f_1 is well defined and it follows from the definition that f_1 is continuous on $[0,1]$.

Let p be a positive integer greater than 1 . Assume f_{p-1} has been defined, then define f_p as follows:
$f_p(x) = f_{p-1}(x)$ if $x \notin [x_n^{j,p+1}, x_n^{j,p}] \cup [y_n^{j,p}, y_n^{j,p+1}]$, where $n = 2,3,\ldots$, $1 \leqq j \leqq 2^{n-1}$. $f_p(x) = g(x)$ if $x \in [x_n^{j,p+1}, x_n^{j,p}]$ where g is a $\surd\!\wedge$ - function of amplitude p . Also $f_p(x) = g(x)$, if $x \in [y_n^{j,p}, y_n^{j,p+1}]$, where g is a $\surd\!\wedge$ - function of amplitude p .

We will show that $\lim_p f_p(x)$ exists for each $x \in I$. This follows, since given $\epsilon > 0$, $x \in I$, $x \neq 0$ there exists a positive integer $N_o^{x,\epsilon}$ such that for all $n,m \geqq N_o^{x,\epsilon}$, $f_m(x) = f_n(x)$ hence $|f_m(x) - f_n(x)| < \epsilon$. To show f is continuous on K^c , where $f(x) = \lim_p f_p(x)$. Since $x \in G_n$ for some n , we have $x \in [x_n^{j,p}, x_n^{j,p-1})$, or $x \in (y_n^{j,p-1}, y_n^{j,p}]$ or $x \in (x_n^{j,1}, y_n^{j,1})$ for some n, j and p . Hence $f(x) = f_m(x)$ for all $m \geqq p$. We will now show f is discontinuous at each point of K . Let $x \in K$. Then $f(x) = 0$ since $f_n(x) = 0$ for all n . If S_x is any open interval containing x then for some integer M there exists $y \in S_x$ such that $f(y) > M$; for if not, $S_x - \{x\}$ would be contained in the family of sets $[x_n^{j,p} , x_n^{j,p-1}] \cup [y_n^{j,p-1}, y_n^{j,p}]$, where $p < M$ and $n = 1,2,\ldots$, $1 \leqq j \leqq 2^{n-1}$. But this would contradict the fact that x is a limit point of K^c . Finally, we show that f is a connectivity function. It follows from the construction of f that f is of Baire Class I. So by Corollary 2 to VII of Section B it is sufficient to show that f is peripherally continuous, but this is clear from the construction of f .

In the preceding example we constructed a connectivity function $f : I \to R$ such that f is constant on K and continuous on K^c. We will now construct an example of a connectivity function f from I onto R which takes on every real value on every open set which contains points of the Cantor set, K and f will be constant on K^c. Although the points of I where f is discontinuous is a proper subset of the Cantor set, f is not of Baire Class I. Of course f is peripherally continuous.

EXAMPLE II. Let $\{G_n\}_{n=1}^{\infty}$ be a base of open intervals for the usual topology on I. If $G_n \cap K \neq \Phi$, let E_n be a perfect subset of $G_n \cap K$, for $n = 1,2,\ldots$. Now let E_n^1 be a perfect subset of E_n such that E_n^1 does not contain an end point of any of the open intervals which were removed to form K. This is possible since every perfect subset of R contains $2^{\aleph_0}$ disjoint perfect subsets. Let T_n be a one to one mapping from E_n^1 onto B, where B is the family of Baire Class I functions from I into R. Define $f : I \to R$ as follows:

$$f(x) = 0 \text{ , if } x \notin E_n^1 \quad \text{for any } n \text{ .}$$
$$f(x) = h(x) \text{ , if } x \in E_n^1 \quad \text{where } T_n(x) = h \text{ .}$$

To show f is a connectivity function. If f is not connectivity this implies there exists some interval $I^1 \subset I$ such that if g is the graph function associated with f, then $g(I^1)$ is not connected. Now by the construction of the E_n^1's, there must exist some n such that $E_n^1 \subset I^1$. It therefore follows that every function of Baire Class I agrees with f at some element $x \in I^1$. Hence by Theorem I, Section C, Chapter II $g(I^1)$ cannot be disconnected. Hence f is a connectivity function.

An interesting consequence of the construction used in Example II is the following: given any perfect set $P \subset R$, then there exists a function

$f : R \to R$ such that f is constant on P^c, taken on every real value on P and f is a connectivity function.

We now state our first main result of this section.

I <u>THEOREM</u> Let Y be any metric space such that $Y = \bigcup_{n=1}^{\infty} F_n$, where $F_n \subset F_{n+1}$ for $n = 1,2,\ldots$, and each F_n is compact, locally connected and connected. If $K \subset I$ is any closed set of first category then there exists a function $f : I \to Y$ such that f is onto, connectivity and the set of points where f is discontinuous is identical to K.

<u>Proof</u>: Since K is closed and of first category, $K^c = \bigcup_{n=1}^{\infty} G_n$, where $G_n \cap G_m = \Phi$ for $n \neq m$, $G_n = (a_n, b_n)$ and some G_n contains at least one of the end points 0 or 1. If $x \in K$ and if N is a neighbourhood of x, then N contains an open interval contained in some G_n. We may also assume that if $G_n = (a_n, b_n)$, $a_n, b_n \in K$. Let m_n be the mid points of (a_n, b_n) for $n = 1,2,\ldots$. If 0 or $1 \notin G_n$ then

let x_n^1 be the midpoint of the open interval (a_n, m_n),

let x_n^p be the midpoint of the open interval (a_n, x_n^{p-1}),

let y_n^1 be the midpoint of the open interval (m_n, b^n),

and y_n^p be the midpoint of the open interval (y_n^{p-1}, b_n).

If $0 \in G_n$ we need only construct the sequence $\{y_n^p\}_{p=1}^{\infty}$. Similarly, if $1 \in G_n$ we need only construct the sequence $\{x_n^p\}_{p=1}^{\infty}$. Then $\lim_p x_n^p = a_n$ and $\lim_p y_n^p = b_n$ for $n = 1,2,\ldots$. For $m = 1,2,\ldots$ construct a mapping f_m from I into Y as follows:

Define f_1 on $[x_n^2, x_n^1]$, $[y_n^1, y_n^2]$ and $[x_n^1, y_n^1]$ such that f_1 maps each of these intervals continuously onto F_1 and $f(x_n^2) = f(x_n^1) = f(y_n^1) = f(y_n^2)$ equals some fixed y_o . This is possible by Hahn-Mazurkiewicz Theorem [See 14], do this for $n = 1,2,\ldots$. If $x \notin [x_n^2, x_n^1] \cup [y_n^1, y_n^2] \cup [x_n^1, y_n^1]$, for any n , then put $f_1(x) = y_o$. It follows that f_1 is continuous on I . Let p be a positive integer > 1 . Assume $f_{p-1}(x)$ has been defined for all $x \in I$. Now define $f_p : I \to Y$ as follows:

$$f_p(x) = f_{p-1}(x) \text{ if } x \notin [x_n^{p+1}, x_n^p] \cup [y_n^p, y_n^{p+1}] ,$$

$$\text{for } n = 1,2,\ldots$$

f_p is defined on $[x_n^{p+1}, x_n^p]$ and $[y_n^{p+1}]$, for $n = 1,2,\ldots$, such that for each fixed n, f_p maps $[x_n^{p+1} , x_n^p]$ and $[y_n^p , y_n^{p+1}]$ onto F_p in such a way that f_p is continuous on each of these intervals and $f_p(x_n^{p+1}) = f_p(x_n^p) = f_p(y_n^p) = f_p(y_n^{p+1}) = y_o$, for $n = 1,2,\ldots$. It is immediately from the definition of f_p , that f_p is continuous and well defined for $p = 1,2,\ldots$. To show that $\{f_p\}_{p=1}^{\infty}$ converges pointwise to a function f . Let $x \in K$, then $f_m(x) = y_o$ for all m . If $x \in K^c$, then $x \in [m_n, b_n)$ or $x \in (a_n, m_n]$ for some n . If $x \in [x_n^{p+1}, x_n^p]$ or $x \in [y_n^p, y_n^{p+1}]$ for some n and some p , then in either case, $f_s(x) = f_t(x)$ for all $s,t \geqq p+1$. Hence $\lim_p f_p(x)$ exists and is equal to $f(x)$ for each $x \in I$. Next we show that f is continuous on K^c and discontinuous at each point of K . That f is continuous on K^c is immediate, since K^c is open and $f(x) = f_p(x)$ for all integers p greater than some fixed integers p_o depending upon x . Let $x \in K$, then we will show that $f(x)$ is mapped onto y_o . Every open interval containing x contains some subinterval I_n of G_n for some n . Then given an open shpere S_{y_o} containing y_o there exists $y \in I_n$ such that $f(y)$ is mapped outside of S_{y_o} . Hence f

is not continuous at x. It is clear that f maps I onto Y.

We finally see that f is a connectivity mapping by applying Corollary 2 of VII in the preceding section. It is clear that f is Baire Class I and peripherally continuous, hence the required result follows by the aforementioned corollary.

Our next result deals with the existence of connectivity function from I onto Y, where Y is a more general space then that considered in I. In this case we will lose the property that f is continuous when restricted to a fairly "large" subset of I. F.D. Whitefield [29] showed in his Ph.D. thesis the existence of a connectivity function f from R into R which is everywhere discontinuous. We were able to change his definition for the function f and by modifying his proof we obtain the following result:

II <u>THEOREM</u> Let I be the unit interval. Let Y be any connected topological space such that $I \times Y$ is completely normal and the topology T on $I \times Y$ has cardinality equal to the cardinality of the continuum. Then there exists a connectivity function f from I onto Y.

<u>Proof</u>: Let $\{G_n\}_{n=1}^{\infty}$ be a base of open sets for the usual topology on I. For each n, select $E_n \subset G_n$ such that E_n is a perfect set, int $E_n = \Phi$ and $E_n \cap E_m = \Phi$ for $n \neq m$. This is possible by II, Section A, Chapter I. By a result mentioned earlier (See IV, Section A, Chapter I) by Kuratowski and Sierpinski, each E_n contains c disjoint perfect set, where c is the cardinality of the continuum. So let $B_n = \{B_r^n\}_{r \in R}$ be a family of perfect sets such that each $B_r^n \subset E_n$, for $r \in R$, and $B_r^n \cap B_s^n = \Phi$ for $s \neq r$. Let A_r be a set which consists of an element from each perfect

set of B_n for $n = 1,2,\dots$. Now put $A = \{A_r\}_{r \in R}$, then A has cardinality equal to the cardinality of the continuum and if $r,s \in R$ and $r \neq s$ we may assume that the selection has been made such that $A_r \cap A_s = \Phi$. Let g be a one to one mapping of the topology T on $I \times Y$ onto A . Let y_o be an arbitrary fixed element of Y . Define $f : I \to Y$ as follows:

$$f(x) = y_o \quad \text{if} \quad x \notin A_r \quad \text{for any} \quad r \in R .$$

$$f(x) = y_o \quad \text{if} \quad x \in A_r \quad \text{for some} \quad r \in R$$
$$\text{and} \quad (\{x\} \times Y) \cap g^{-1}(A_r) = \Phi .$$

Finally suppose $x \in A_r$ for some $r \in R$ and $(\{x\} \times Y) \cap g^{-1}(A_r) \neq \Phi$. Put $U = g^{-1}(A_r)$. If $\{y | (x,y) \in U\} = Y$, then put $f(x) = y_o$. If $\{y | (x,y) \in U]$ is a proper subset of Y , then this set is open in Y , because for every $y \in \{y | (x,y) \in U\}$ there exists an open sets V_x and U_y containing x and y respectively such that $V_x \times U_y \subset U$. Let $p \in \cup V_y$, where the union is taken over all $y \in \{y | (x,y) \in U\}$, then $(x,p) \in U$, so $\{y | (x,y) \in U\}$ is open in Y . Since Y is connected there exists a boundary point y^1 of $\{y | (x,y) \in U\}$ such that $y^1 \notin \{y | (x,y) \in U\}$. Let W be an arbitrary neighbourhood of (x,y^1) in $I \times Y$. Then choose open sets U_x and V_{y^1} containing x and y^1 respectively such that $U_x \times V_{y^1} \subset W$, since y^1 is a limit point of $\{y | (x,y) \in U\}$, it follows there exists (x,y) such that $y \in \{y | (x,y) \in U\}$ and $(x,y) \in U_x \times V_{y^1}$. Hence (x,y^1) is a boundary point of U and $(x,y^1) \notin U$. Put $f(x) = y^1$.

To show f has the required properties. Suppose f is not a connectivity function. Let h be the graph function of f . Then there exists some connected set $C \subset I$ such that $h(C)$ is not connected in

$I \times Y$. Since $I \times Y$ is completely normal there exists open sets U and V such that $h(C) \subset U \cup V$ and $U \cap V = \Phi$. Let A_U and A_V denote the respective images of U and V under g . By construction $A_U \cap C \neq \Phi$ and $A_V \cap C \neq \Phi$. Let $x \in A_U \cap C$ and suppose $f(x) \neq y_o$. Then by the definition of f , $(\{x\} \times I) \cap U \neq \Phi$ and $\{y | (x,y) \in U\} \neq Y$. Hence $(x,f(x))$ is a limit point of U and $(x,f(x)) \notin U$. Clearly $(x,f(x)) \notin V$. But his contradicts $h(C) \subset U \cup V$, hence $f(x) = y_o$ for all $x \in A_U \cap C$. Similarly if $x \in A_V \cap C$ then $f(x) = y_o$.

Let U_1 and V_1 denote the projection of U and V onto I . Then U_1 and V_1 are open in I and $U_1 \cup V_1$ contain C since $U \cup V$ contains $h(C)$. Also $U_1 \cap V_1$ contains a point of C . It follows from the construction of a that $A_U \cap C$ and $A_V \cap C$ are dense in C . Hence $U_1 \cap V_1$ contains points of $A_U \cap C$ and $A_V \cap C$. Let $x \in (U_1 \cap V_1) \cap (A_U \cap C)$ then $f(x) = y_o$ and since $x \in U$, $(\{x\} \times Y) \cap U \neq \Phi$. Let G be an open set in $I \times Y$ containing $(x,y_o) = (x,f(x))$. Then there exists an open set $W \subset I$ such that $x \in W$ and an open set J containing y_o such that $W \times J \subset G$. Since $f(x) = y_o$ if follows from the definition of f that either y_o is a boundary point of $\{y | (x,y) \in U\}$ or $Y = \{y | (x,y) \in U\}$. In either case their exists some $y \in J$ such that $y \neq y_o$ and $(x,y) \in U$. Hence (x,y_o) is a limit point of U . Also $(x,y_o) \in U$ since $h(C) \subset U \cup V$ and $U \cap V = \Phi$ and U,V are open. Similarly, if $x \in (U_1 \cap V_1) \cap (A_V \cap C)$, then $f(x) = (x,y_o) \in V$.

Now let $x \in (U_1 \cap V_1) \cap (A_V \cap C)$, then $(x,f(x)) \in V$ by the above. Hence there exists an open set G containing $(x,f(x))$ such that $G \subset V$ and $G = W \times J$, where W,J are open sets containing x and $f(x)$

respectively, and W is choosen such that $W \subset U_1 \cap V_1$. Since $A \cap C$ is dense in C, W contains a point z of $A_U \cap C$. But $z \in (U_1 \cap V_1) \cap (A_U \cap C)$ implies $(x,f(x)) \in U$. This contradicts the fact that U and V are disjoint. Hence the graph of f is connected.

Finally we show that f maps I onto Y. Let $y \in Y$ and consider the open set $H_y = Y - \{y\}$. $I \times H_y$ is open in $I \times Y$ and let A_r be the image of $I \times H_y$ under g. Then if $x \in A_r$ it follows from the definition of f that $f(x) = y$. Hence f is onto.

It is readily seen from the definition of A in the proof of the last theorem that if C is any connected subset of I then $C \subset \overline{A_r \cap C}$ for every $r \in R$. We will now show that if I^1 is a non-degenerate subinterval of I, then $f(I^1) = Y$. For let $y \in Y$ and $H_y = Y - \{y\}$. Let $g(I \times H_y) = A_r$, then by the construction of A_r, there exists some $x \in I^1$ such that $x \in A_r$. It follows from the construction of f that for all $x \in A_r$, $f(x) = y$. Hence $f(I^1) = Y$.

Does there exists a family A of subsets of $I \times I = I^2$ such that $A = \{A_r\}_{r \in R}$, where each A_r has cardinality equal to the cardinality of the continuum, $A_r \cap A_s = \phi$ for $r \neq s$, with the property that if C is any connected subset of the plane then $A_r \cap C$ is dense in C for every $r \in R$? Although such a family A of subsets of I (or R) exists, no such family exists in I^2. In order to show this we will need a theorem by Cornette [2]. Let E denote the set in a plane with a dispersion point e. That is, E is connected and $E-e$ is totally disconnected (see [14]).

III THEOREM (Cornette [2] Theorem 2). There does not exist a connectivity function f with domain $I \times I = I^2$ and range E.

If there exists a family A of subsets of I^2 which has the property outlined above, then by using the same construction as in II we can construct a connectivity function f from I^2 onto Y, where $I^2 \times Y$ is completely normal. However $I^2 \times E$ is completely normal and hence the existence of the family A would contradict III. Hence no such family exists in I^2.

D. THE EXISTENCE OF LARGER CONNECTED TOPOLOGIES FOR A TOPOLOGICAL SPACE (X,T).

In this section we present a few results concerning the existence of a topology T^* for a connected topological space (X,T) such that $T^* \supset T$ and (X,T^*) is a connected topological space. Let Y be the set consisting of the two elements 0 and 1. Let σ be the topology on Y where the open sets are ϕ, $\{0\}$ and Y.

I <u>THEOREM</u> Let (X,T) be any connected topological space. There exists a topology T^* on X such that $T^* \supset T$ and (X,T^*) and (X,T) have the same connected sets if and only if there exists a discontinuous connectivity function from (X,T) onto (Y,σ).

<u>Proof</u>: Suppose there exists a discontinuous connectivity function f from (X,T) onto (Y,σ). Then there exists some open set V in Y such that $f^{-1}(V)$ is not open in T. Put T^* equal to the topology generated by T and $f^{-1}(V)$. Clearly T^* is larger than T and (X,T) and (X,T^*) have the same connected sets by Theorem 2.3 [7].

Conversely, suppose there exists a topology T^* on X such that $T^* \supset T$ and (X,T) and (X,T^*) have the same connected sets. To construct

a discontinuous connectivity function from (X,T) onto (Y,σ). Let $A \in T^*$ such that $A \notin T$. Put $f(A) = 0 \in Y$ and if $x \in X$ and $x \notin A$ put $f(x) = 1$. Then $f : (X,T) \to (Y,\sigma)$ in discontinuous. But $f : (X,T^*) \to (Y,\sigma)$ is continuous and hence a connectivity function. Since $T^* \times \sigma \supseteq T \times \sigma$, f is a connectivity function from (X,T) onto (Y,σ).

II <u>THEOREM</u> If (X,T) is connected, Hausdorff and first countable, then there exists a strictly larger topology T^* on X such that the connected sets of (X,T^*) and (X,T) are identical.

<u>Proof</u>: Let $x_o \in X$ and let $\{B_n\}_{n=1}^{\infty}$ be a monotone decreasing local neighbourhood base of open sets at x_o. Let $x_1 \in B_1$ such that $x_1 \neq x_o$. Choose V_{x_1} and $V^1_{x_o}$, open neighbourhoods of x_1 and x_o respectively such that $V_{x_1} \cap V^1_{x_o} = \phi$ and $V^1_{x_o} \subset B_2$. Select $x_2 \in V^1_{x_o}$ such that $x_2 \neq x_o$. In general suppose x_k has been choosen such that $x_k \in B_k$. Then let V_{x_k} and $V^k_{x_o}$ be open and disjoint neighbourhoods of x_k and x_o respectively such that $V^k_{x_o} \subset B_{k+1}$. Let $x_{k+1} \in V^k_{x_o}$ such that $x_{k+1} \neq x_o$. Hence we have selected a sequence $P = \{x_k\}_{k=1}^{\infty}$ such that $x_k \to x_o$. If follows from the way we selected the x_k, for $k = 1,2,...$ and from the fact that (X,T) is connected that P is not open. Let T^* be the topology generated by P^c and the topology T.

Suppose (X,T^*) is not connected. Then $X = A_1 \cup A_2$ where $A_1 \cap A^2 = \phi$ and A_1 and A_2 are open in T^*. If $P \subset A_1$, then for every element $p \in P$ there exists an open set N^*_p in T^* and containing

p such that $N_p^* \subset A_1$. Since for every open neighbourhood N_p of p in T, $N_p \cap P^c$ is not a neighbourhood of p, $N_p^* = N_p \cup B$, where $B \subset P^c$. Hence there exists an open neighbourhood N_p of p in T such that $N_p \subset A_1$. If $x \in A_1$ and $x \notin p$, the neighbourhood of x in T^* are formed from those of x in T by deleting the points of P. But since $P \subset A_1$, it follows there exists an N_x open in T such that N_x is contained in A_1. Hence A_1 is open in T. Also $x_o \in A_1$; since for each $x_n \in P$, choose $y_n \in V_{x_n} \cap N_{x_n}$, where $N_{x_n} \subset A_1$ and V_{x_n} are as constructed earlier, then $y_n \to x_o$ in the T^* topology, hence $x_o \in A_1$.

A_2 is open in T. This follows since given any neighbourhood N_x^* of $x \in A$ such that $N_x^* \subset A_2$, we can find a nbhd N_x of x, open in T such that $N_x \cap P = \Phi$, since X is Hausdorff and hence $N_x \subset A_2$.

So $A_2 \cap P \neq \phi$ and $A_1 \cap P \neq \Phi$. If $x_o \in A_1$, then all $x_n \in P$, except for a finite number, must also be in A_1, for if an infinite subsequence of P say $x_{n_k} \in A_2$, for $k = 1,2,\ldots$, then we can select $y_{n_k} \in V_{x_{n_k}} \cap N_{x_n}$, where $N_{x_n} \subset A_2$ such that $y_{n_k} \to x_o$ is the T^* topology which contradicts A_1 and A_2 being open and disjoint in T^*. Now we can show as before that A_1 and A_2 are open in T. A contradiction. Hence (X,T^*) is connected.

Next we show that (X,T^*) and (X,T) have the same connected sets. Let $f : X \to (Y,\sigma)$ where $f(P^c) = 0$ and $f(P) = 1$, and Y is as in I. Then f is continuous in the T^* topology and hence a connectivity function from (X,T^*) onto (Y,σ). Hence f is discontinuous and connectivity in the T topology. Now apply I again and we see that (X,T^*) and (X,T)

have the same connected sets.

We next give an example to show that if Hausdorff were omitted from the hypothesis the conclusion of the above theorem no longer holds.

EXAMPLE I. X is an infinite set. Find $x_o \in X$. The only neighbourhood of x_o is the whole space. For $x \in X$ and $x \neq x_o$, let x be open. Let the topology generated by these open sets be denoted by T. (X,T) is connected. However if T^* is any larger topology on X, then (X,T^*) is not connected.

The following definition is well known [22].

III DEFINITION A topological space (X,T) is semi-locally connected if for every $x \in X$ there exists a local base B at x such that for every $V \in B$, $X - V$ has at most a finite number of components.

It is clear that the real line with the usual topology is semi-locally connected. However we have the following result.

IV THEOREM Let T^* be any topology on R such that (R,T^*) is connected and T^* is strictly larger then the usual topology on R. Then (X,T^*) is not semi-locally connected.

Proof: Suppose (R,T^*) is semi-locally connected. Let $f : (R,T) \to (R,T^*)$ be the identity mapping. Then both f and f^{-1} are connected mappings and f^{-1} is continuous. Hence by Theorem 3.5 [22] f is continuous. This contradicts the fact that $T^* \underset{+}{\supset} T$. Hence (R,T^*) is not semi-locally connected.

In concluding we consider the question of constructing a larger topology T^* on a countable connected Hausdorff topological space (X,T)

such that (X,T^*) has the same connected sets as (X,T). All the examples of countable connected Hausdorff spaces which we were able to find in the mathematical literature have a countable base. Hence by using Theorem II we can construct a larger topology T^* on each of these given spaces such that (X,T) and (X,T^*) have the same connected sets. First we will show the existence of a countable connected Hausdorff space (X,T) which has an infinite subset E such that at every point X of E the neighbourhood system of x does not have a countable base. Secondly we shall show that even in this case there exist a topology T^* on X such that $T^* \underset{+}{\supset} T$ and (X,T^*) has the same connected sets as (X,T).

EXAMPLE II. In this example we show the existence of a countable connected Hausdorff space (X,T) which is not first countable at each point x of an infinite subset E of X. This will be a modification of an example of a countable connected Hausdorff space with a dispersion point constructed by Martin [20].

Let $\{C_n\}_{n=0}^{\infty}$ be a countable collection of subsets of rational numbers such that $C_i \cap C_j = \Phi$ for $i \neq j$ and each C_i is dense in the real numbers. Let $X = \{(x,y) \mid y$ is a non negative integer and $x \in C_y\}$. Thus the points of X will sometimes be referred to as lying on the lines in the plane with integer ordinates. Let E be the set of points y_n, for $n = 0, \pm 1, \pm 2, \ldots$, where for each n, y_n is an interior point of the open interval $(n, n+1)$ and $y_n \in C_o$. Now for each $x \in X - E$ we will define the neighbourhood base at x as in [20].

Suppose $p \in X - E$, n is a non negative integer, $z \in C_n$ and $p = (z,n)$, where $z \geq 0$ if $n = 0$. Let k be a positive integer,

δ be a positive number less than $\frac{\pi}{4}$. Let $V_{k\delta}(p)$ be the intersection of $X - E$ and $\{(x,y) \mid y = n$ and $(z + k\pi - \delta) < x < (z + k\pi + \delta)\}$ and let $U_\delta(p) = [\bigcup_{k=1}^{\infty} V_{k\delta}(p) \cup \{p\}]$.

Suppose $p \in X - E$, $z \in c_o$ and n is a negative integer such that $n < z < n+1$, and $p = (z,0)$. Let k be a positive integer and δ a positive number less than $\frac{\pi}{4}$. Let $V_{k\delta}(p)$ be the intersection of $X - E$ and $\{(x,y)\}: 0 \leq y \leqq -n$, and $(z + k\pi - \delta) < x < (z + k\pi + \delta)\}$, and let $U_\delta(p) = [\bigcup_{k=1}^{\infty} V_{k\delta}(p)] \cup \{p\}$.

Suppose $p \in E$. V is an open neighbourhood of p if V consists of an infinite number of horizontal lines from X such that V^c also consists of a infinite number of horizontal lines, $C_o - p \subset V^c$ and for every line in V there exists a finite number of points from this line which are in V^c.

We omit the proof that the above defines a basis for a topology T on X. The space X is countable and the proof that (X,T) is a Hausdorff space is the same as that given in [20].

V <u>LEMMA</u> (X,T) is a connected space.

<u>Proof</u>: Suppose $X = A_1 \cup A_2$ where $A_1 \cap A_2 = \Phi$ and $A_1, A_2 \in T$. We may assume some $y_n \in E$ is an element of A_1. Let $x \in A_2$ such that $x \notin E$. Than there exists a postive number δ such that $U_\delta(x) \subset A_2$. Since A_2 is closed, give a positive integer M there exists $z \in C_o$ such that $z < -M$ and $(z,o) \in A_2$, $z \notin E$. Hence when we consider C_M as a subset of the plane there exists an infinite number of points (p,M) such that $(p,M) \in C_M$ and $(p,M) \in A_2$. Hence $y_n \in A_2$. This contradicts

our assumption. Hence (X,T) is connected.

VI <u>LEMMA</u> If $x \in E$, then the neighbourhood system of x does not have a countable base.

<u>Proof</u>: Suppose $x \in E$ and there exists a decreasing family of open sets $\{B_n\}_{n=1}^{\infty}$ which form a local base at x . From each B_n select a horizontal line P_n in the plane such that $P_n \in X$ and all but a finite subset of P_n is contained in B_n . Let $x_n \in P_n$ such that $x_n \in B_n$. Put $P_n^1 = P_n - \{x_n\}$. Consider the open neighbourhood V of x which consists of the x together with the family of horizontal line segments P_n^1 , for $n = 1,2,\ldots$. Then B_n is not a subset of V for $n = 1,2,\ldots$ Hence the family $\{B_n\}_{n=1}^{\infty}$ cannot form a local base at x .

Finally let $(p,n) \in X$ where $n \neq 0$. Then following the same construction used in II we can construct a topology T^* on X such that $T^* \supset T$ and (X,T^*) is connected.

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